Better Size Estimation for Sparse Matrix Products

Rasmus R. Amossen, Andrea Campagna, Rasmus Pagh

IT University of Copenhagen

September 1, 2010

The problem

Boolean matrix multiplication

The problem

Boolean matrix multiplication

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad \times \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

The problem

Boolean matrix multiplication

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$a^3_{i,j} = \bigvee_{h \in \{1,\ldots,n\}} a^1_{i,h} \wedge a^2_{h,j}.$$

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Boolean matrix multiplication

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$$a^3_{i,j} = \bigvee_{h \in \{1,\ldots,n\}} a^1_{i,h} \wedge a^2_{h,j}.$$

How many true values are there in the resulting matrix?

The problem

Join size estimation

R_1	
a	b
1	1
1	2
3	2
12	3
2	3
2	4

$\bowtie$

R_2	
b	c
1	2
2	2
2	5
3	12
4	12

$=$

$R_{1,2}$		
a	b	c
1	1	2
1	2	2
1	2	5
3	2	2
3	2	5
12	3	12
2	3	12
2	4	12

The problem

Join size estimation

$\pi_{a,c}$

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a	b	c
1	1	2
1	2	2
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12	3	12
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$=$

R	
a	c
1	2
1	2
1	5
3	2
3	5
12	12
2	12
2	12

How many distinct rows?

Equivalence of problems

R_1	
a	b
1	1
1	2
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$\bowtie$

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$$\begin{pmatrix} \quad \end{pmatrix} \times \begin{pmatrix} \quad \end{pmatrix} = \begin{pmatrix} \quad \end{pmatrix}$$

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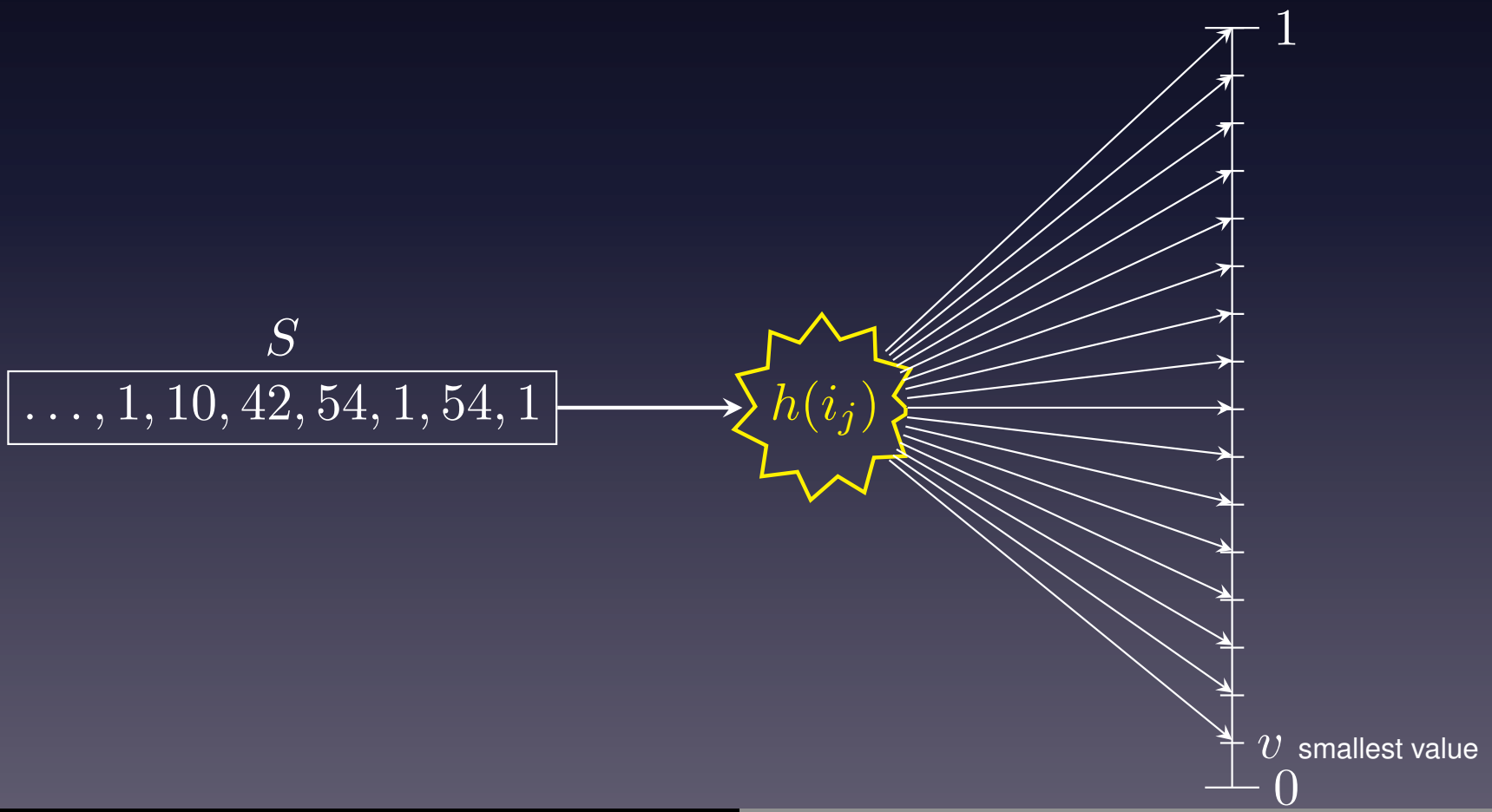
Previous and our results

- **Previous results:** $O(n/\varepsilon^2)$ for any fixed constant error probability (Cohen);
- **Our contribution:** $O(n)$ for any $\varepsilon > 4/\sqrt[4]{n}$ and polynomially small error probability for any fixed ε.

The tools

- S is a stream of elements $i \in [n]$;
- $h : [n] \to (0, 1]$ is a pairwise independent hash function;
- If h distributes the values evenly, then $1/v$ distinct items.



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- **Problem:** large variance;

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- **Solution:**
  - take the k smallest values;

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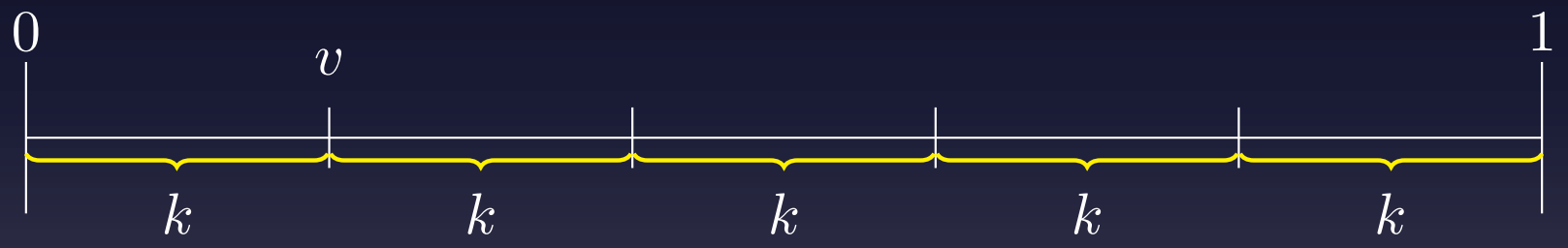
- **Problem:** large variance;
- **Solution:**
  - take the k smallest values;
  - If v is the value of the kth smallest, estimate k/v distinct items.

The tools

- **Problem:** large variance;
- **Solution:** take the k smallest values (Bar-Yossef et al.);



- Density is $v/k \to k/v$ estimated distinct items.

The tools

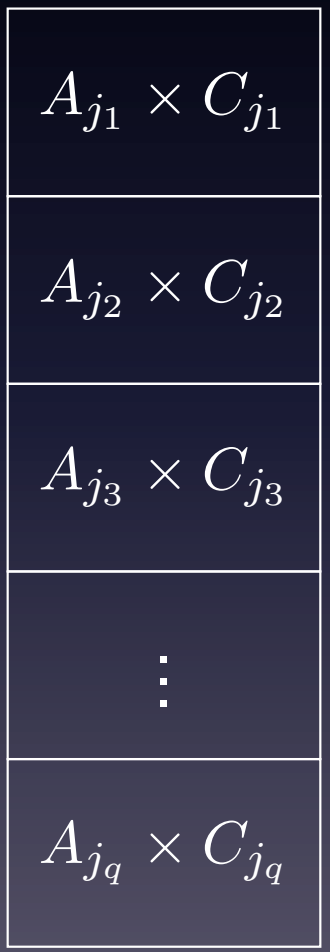
R	
a	c
1	2
1	2
1	5
3	2
3	5
12	12
2	12
2	12

The tools

- **Problem:** find a suitable hash function;
- **Solution:**
  - draw independently at random $h_1 : U \to (0, 1]$ and $h_2 : U \to (0, 1]$ from a family of pairwise independent hash functions;
  - build $h(x, y) = (h_1(x) - h_2(y)) \mod 1$;
  - $h(x, y)$ built in this way is *pairwise independent*.

Notation

$\pi_{a,c}(R_1 \bowtie R_2)$



- $|R_1| + |R_2| = n$;
- $A_j = \{i | (i, j) \in R_1\}$;
- $C_j = \{i | (j, i) \in R_2\}$;
- Note: $\pi_{a,c}(R_1 \bowtie R_2) = \bigcup_j A_j \times C_j$.

Question: how to generate the pairs in $A_j \times C_j$ for any given j?

The algorithm

$R_1 \bowtie R_2 =$

$R_{1,2}$		
a	b	c
2	3	12
1	1	2
1	2	2
12	3	12
1	2	5
3	2	2
2	4	12
3	2	5

The algorithm

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$\longrightarrow$

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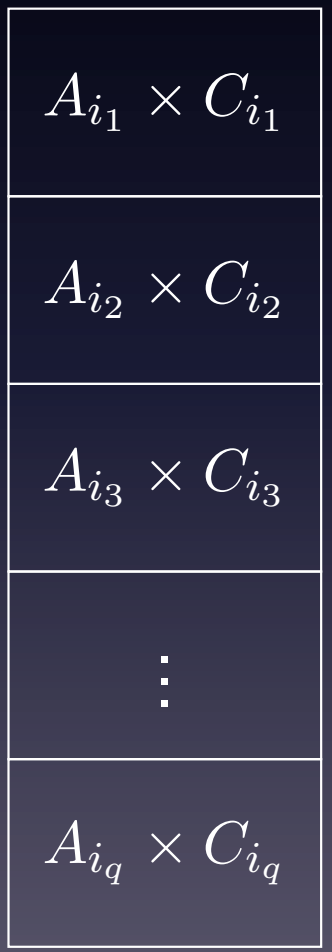
- First step: sort according to b;
- $O(n)$ in the RAM model (e.g. using hashing);
- $sort(n)$ I/Os in the I/O model.

The algorithm

$\pi_{a,b}(R_1 \bowtie R_2)$



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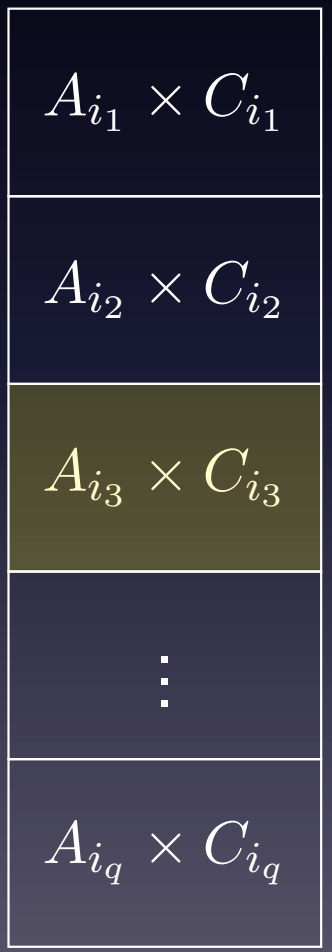

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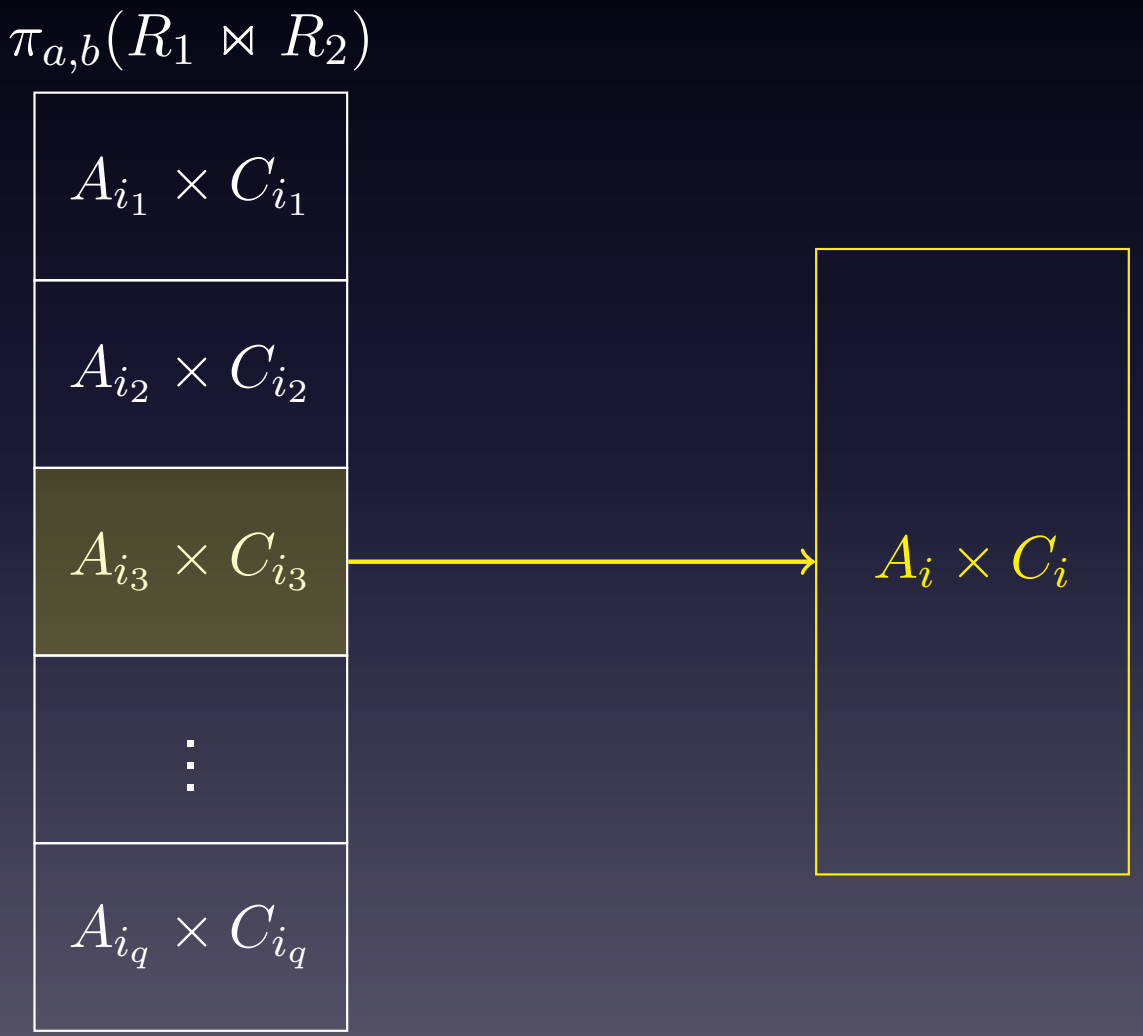

The algorithm

- $h(x, y) := h_1(x) - h_2(y) \mod 1$;
- $Z = \pi_{a,b}(R_1 \bowtie R_2) = \bigcup_i A_i \times C_i$, so we can focus on a single generic $A_i \times C_i$;
- find the k smallest values in $h(A_i \times C_i) = (h_1(A_i) - h_2(C_i)) \mod 1$.

The algorithm

$$m_{x,y} = h(a_{q_x}, c_{q_y}) = h_1(a_{q_x}) - h_2(c_{q_y}) \mod 1.$$

$h_2(c)$

$h_1(a)$

	c_{j_1}	c_{j_2}	c_{j_3}	c_{j_4}	c_{j_5}
a_{q_1}	0.9	0.8	0.7	0.5	0.28
a_{q_2}	0.93	0.83	0.73	0.53	0.31
a_{q_3}	0.02	0.92	0.82	0.52	0.3
a_{q_4}	0.15	0.05	0.95	0.65	0.43
a_{q_5}	0.22	0.12	0.02	0.72	0.5
a_{q_6}	0.52	0.42	0.32	0.02	0.8
a_{q_7}	0.71	0.61	0.51	0.21	0.99
a_{q_8}	0.85	0.75	0.65	0.35	0.13

The algorithm

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$h_2(c)$

$h_1(a)$

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threshold p

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$h(x_{i-1}, y)$
$\vee$
$h(x_i, y_j)$

The algorithm

$$m_{x,y} = h(a_{q_x}, c_{q_y}) = h_1(a_{q_x}) - h_2(c_{q_y}) \mod 1.$$

$h_2(c)$

	c_{j_1}	c_{j_2}	c_{j_3}	c_{j_4}	c_{j_5}
a_{q_1}	0.9	0.8	0.7	0.5	0.28
a_{q_2}	0.93	0.83	0.73	0.53	0.31
a_{q_3}	0.02	0.92	0.82	0.52	0.3
a_{q_4}	0.15	0.05	0.95	0.65	0.43
a_{q_5}	0.22	0.12	0.02	0.72	0.5
a_{q_6}	0.52	0.42	0.32	0.02	0.8
a_{q_7}	0.71	0.61	0.51	0.21	0.99
a_{q_8}	0.85	0.75	0.65	0.35	0.13

$h_1(a)$

$0.9 > 0.85$

The algorithm

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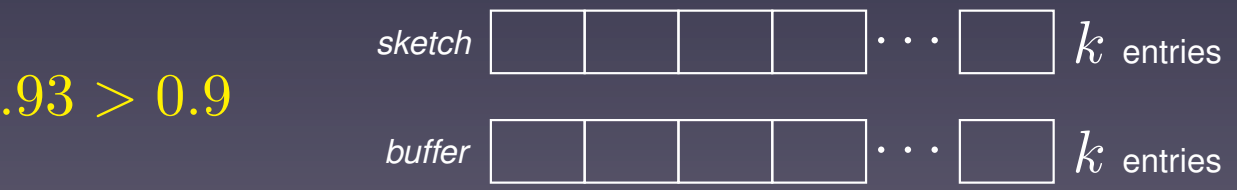
a_{q_8}	0.85	0.75	0.65	0.35	0.13

$h_1(a)$

$0.93 > 0.9$



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$h_1(a)$

$0.02 < 0.93$

The algorithm

$$m_{x,y} = h(a_{q_x}, c_{q_y}) = h_1(a_{q_x}) - h_2(c_{q_y}) \mod 1.$$

$h_2(c)$

$h_1(a)$

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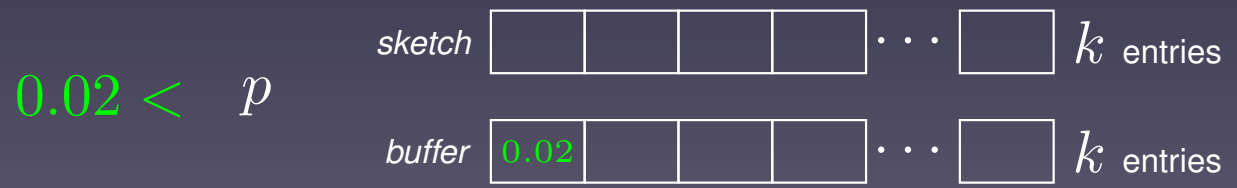
$0.02 < p$



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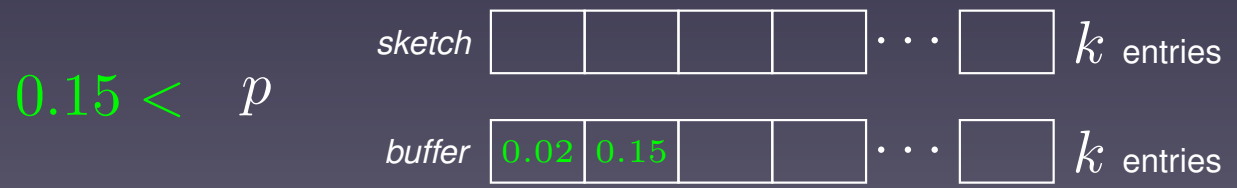
$0.15 < p$



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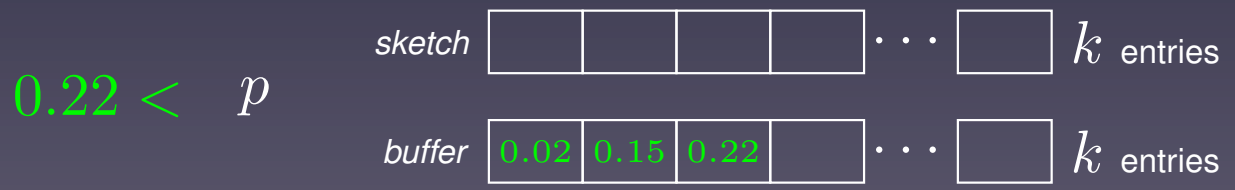
$0.22 < \quad p$



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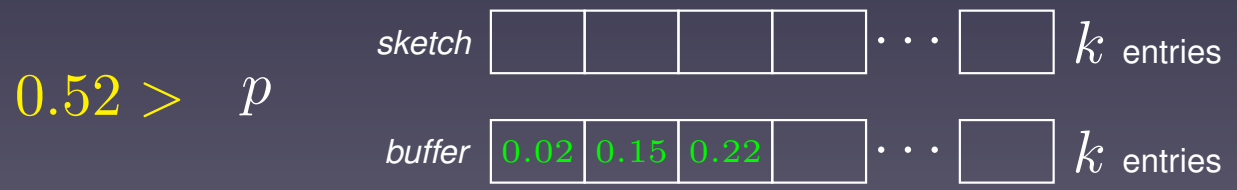
a_{q_8}	0.85	0.75	0.65	0.35	0.13

$0.52 > \quad p$



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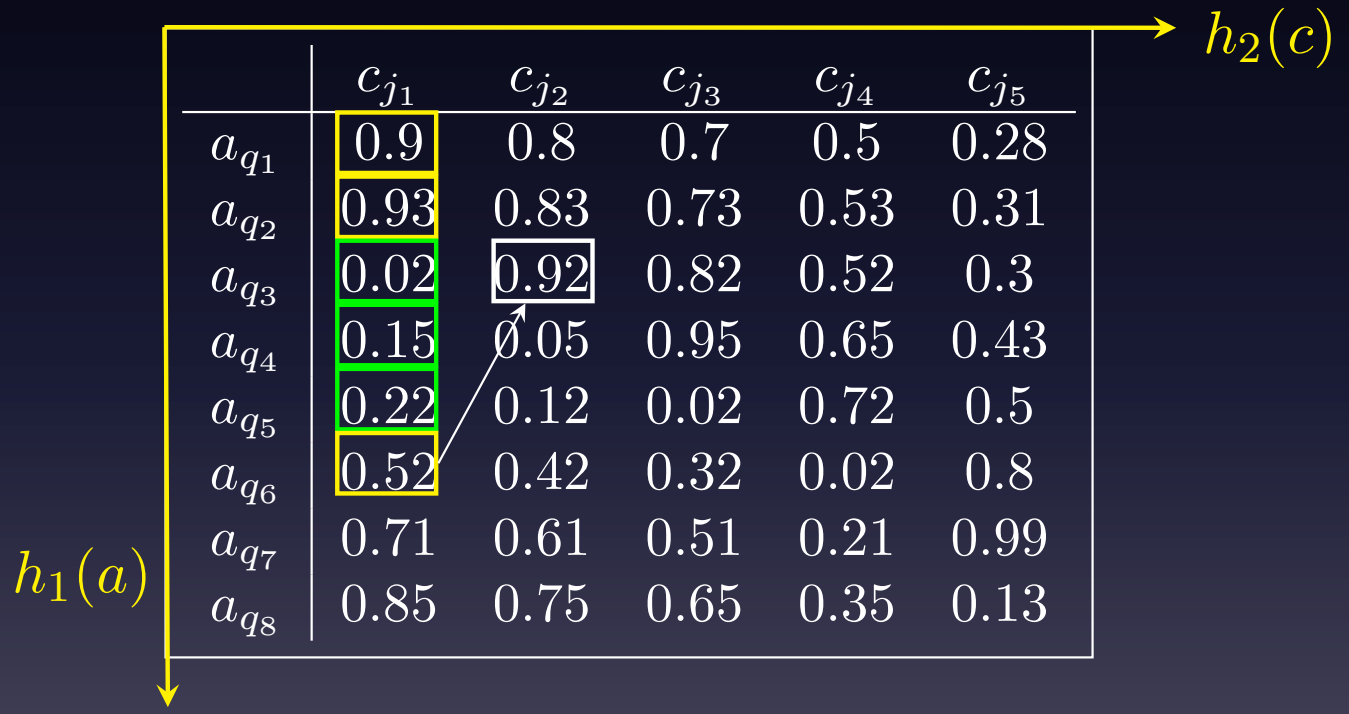

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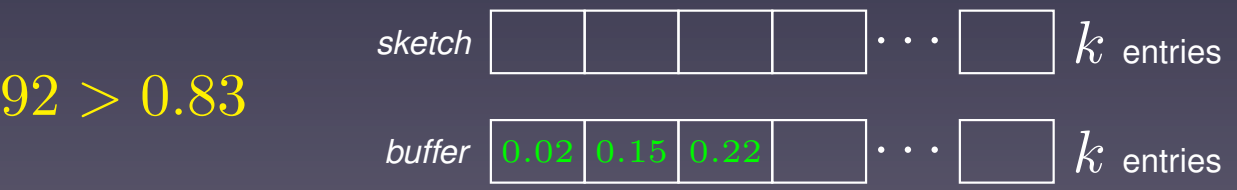
$0.92 > 0.83$



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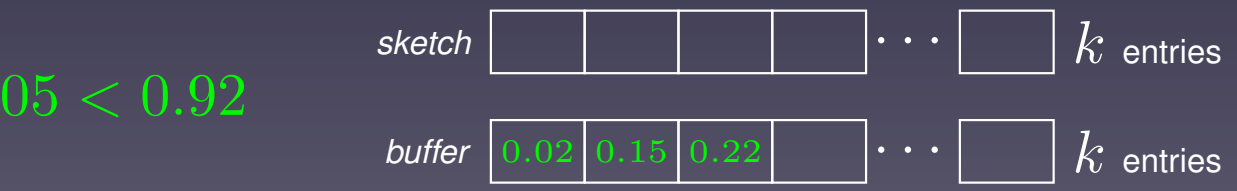
$0.05 < 0.92$



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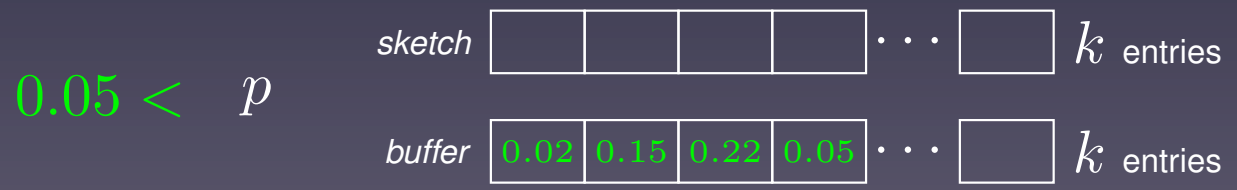
$0.05 < p$



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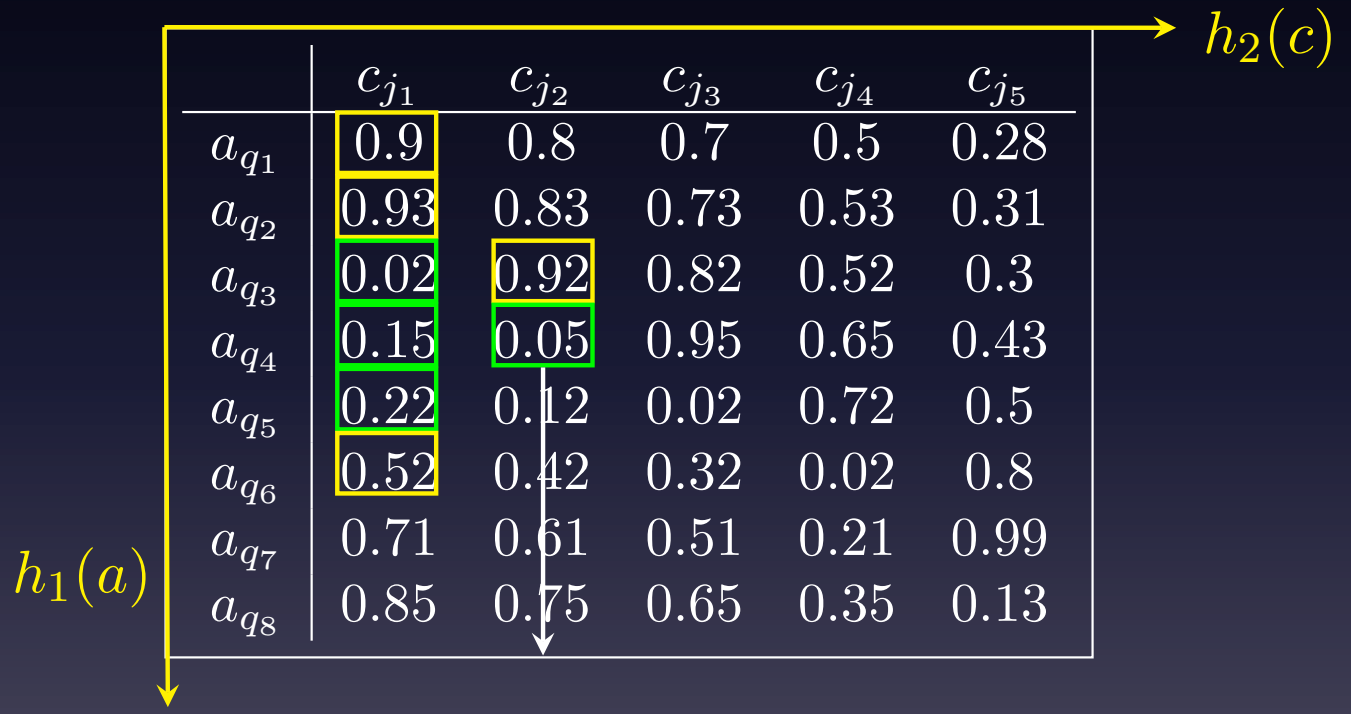

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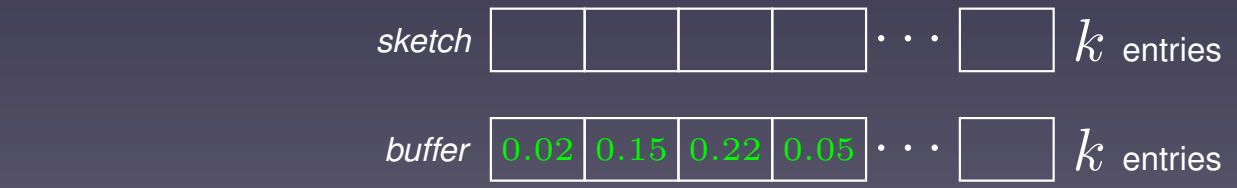

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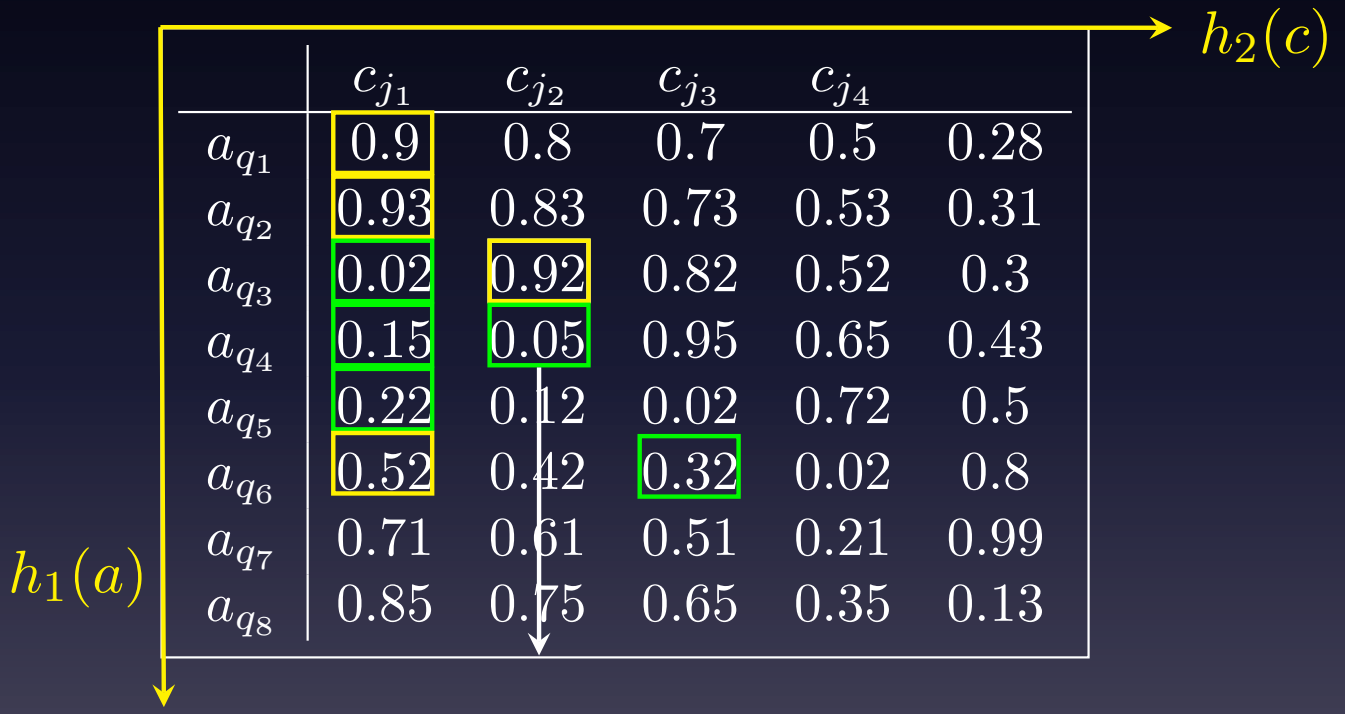
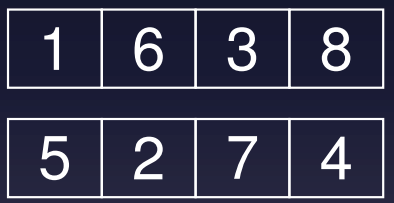


Not sorted

The algorithm (Buffers merging)



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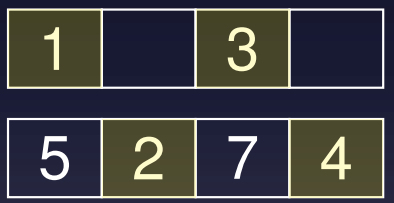

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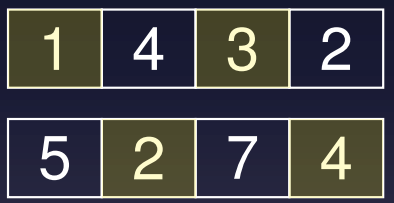

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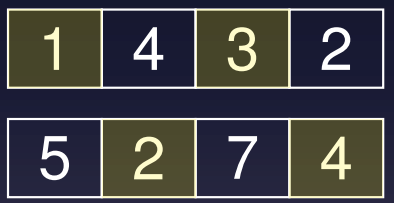

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- Note: *no need to keep the two structures sorted!*

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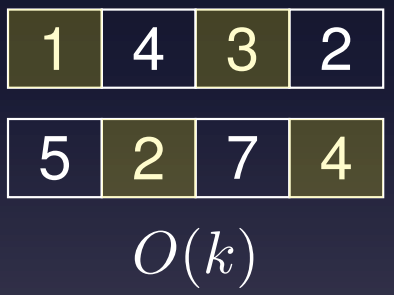

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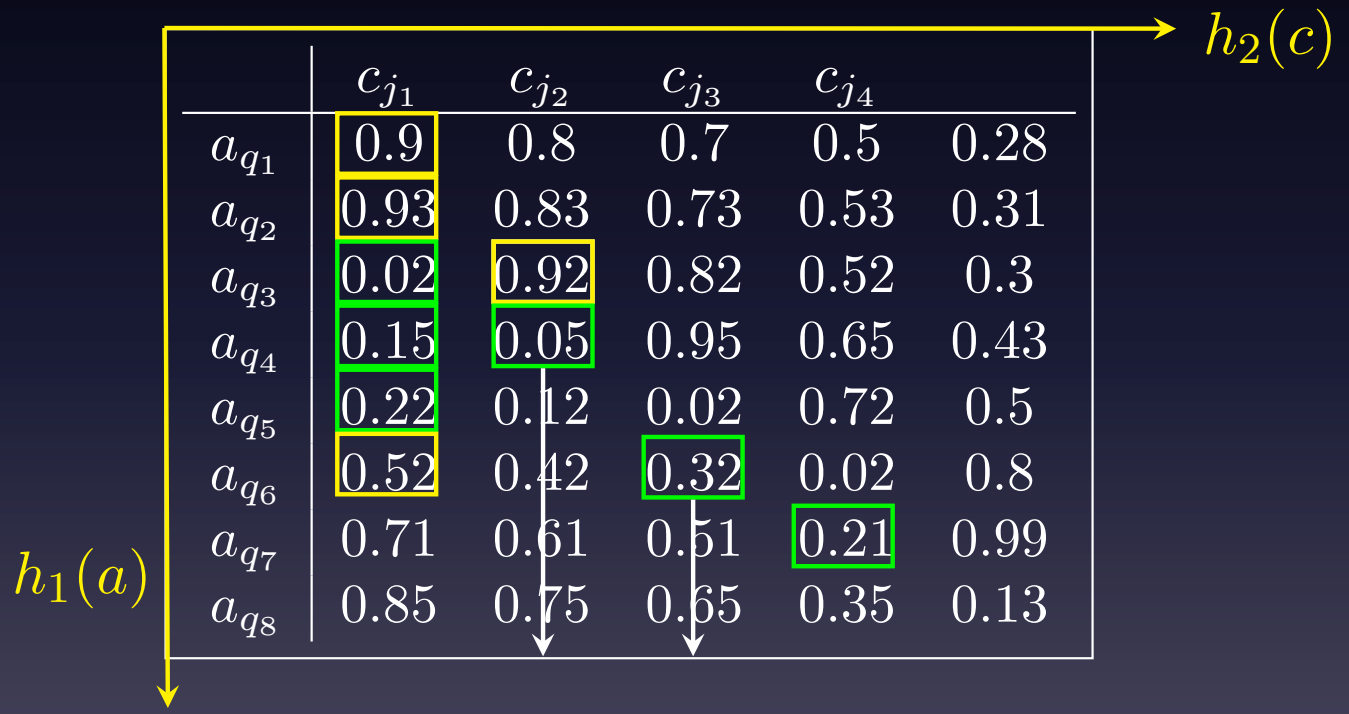

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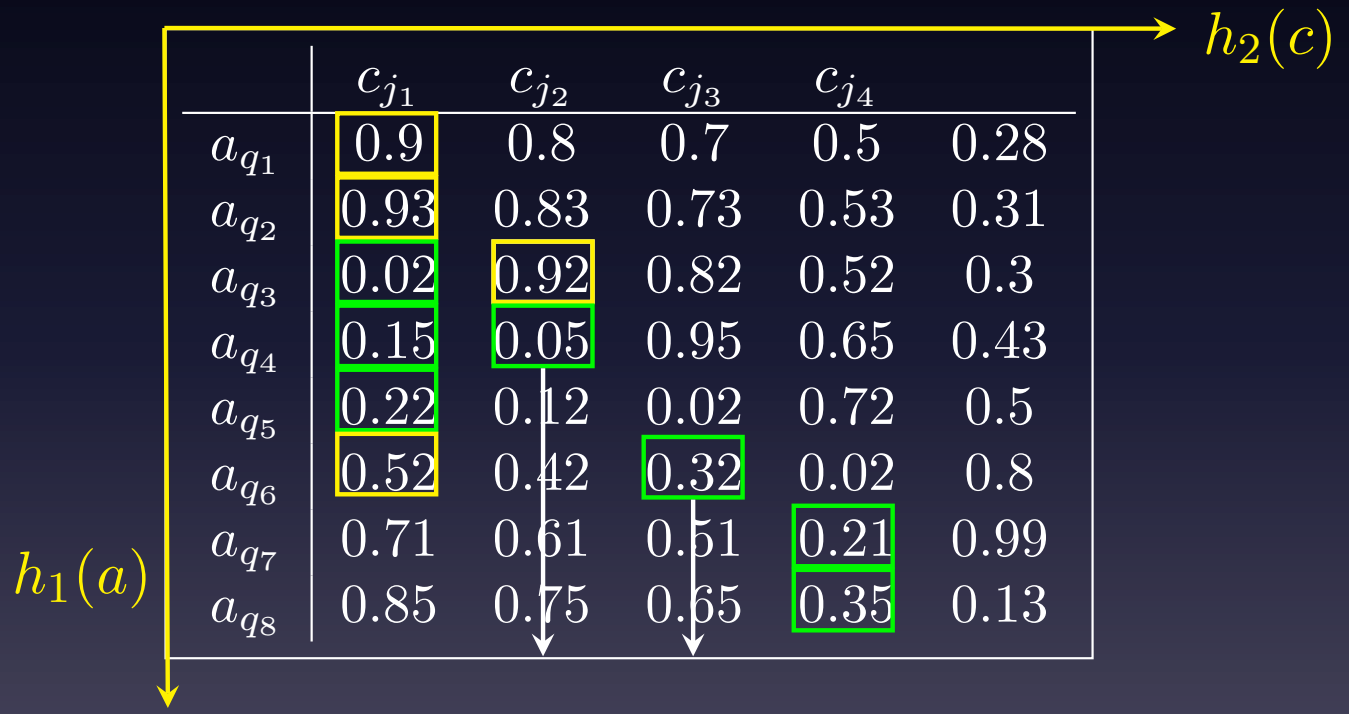

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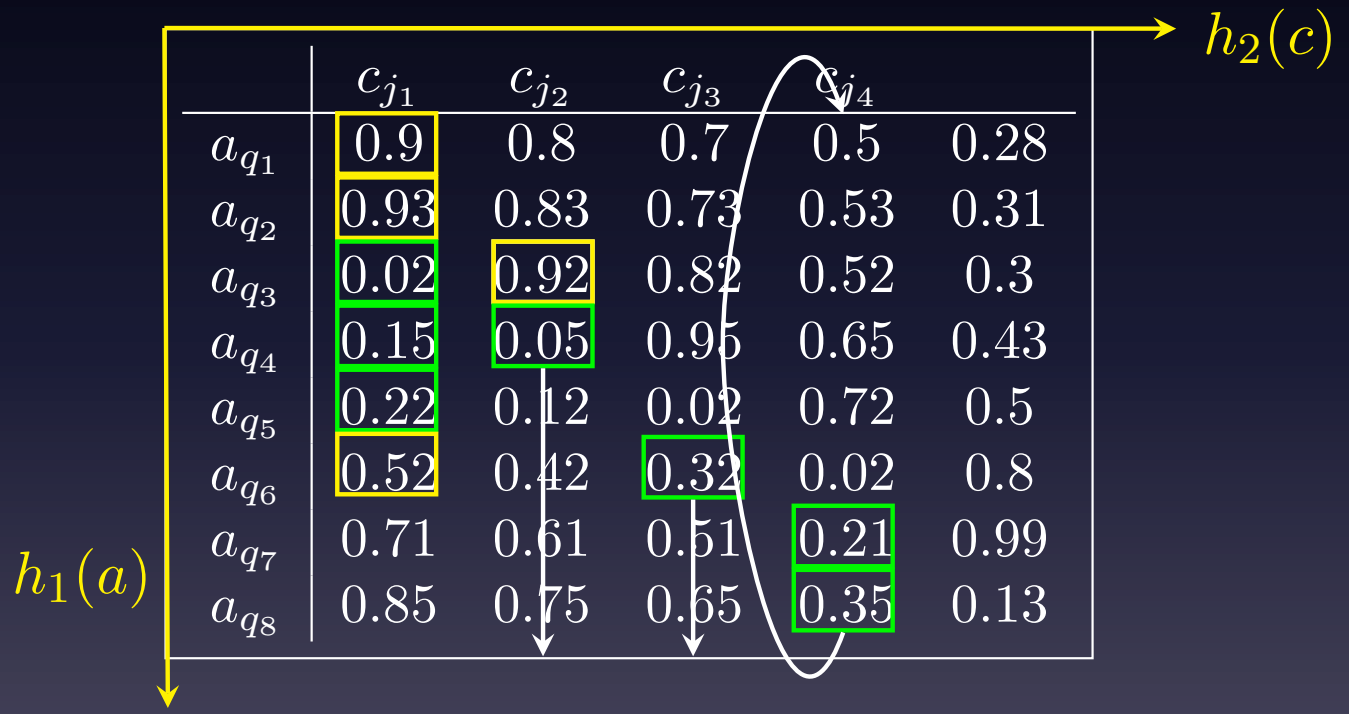

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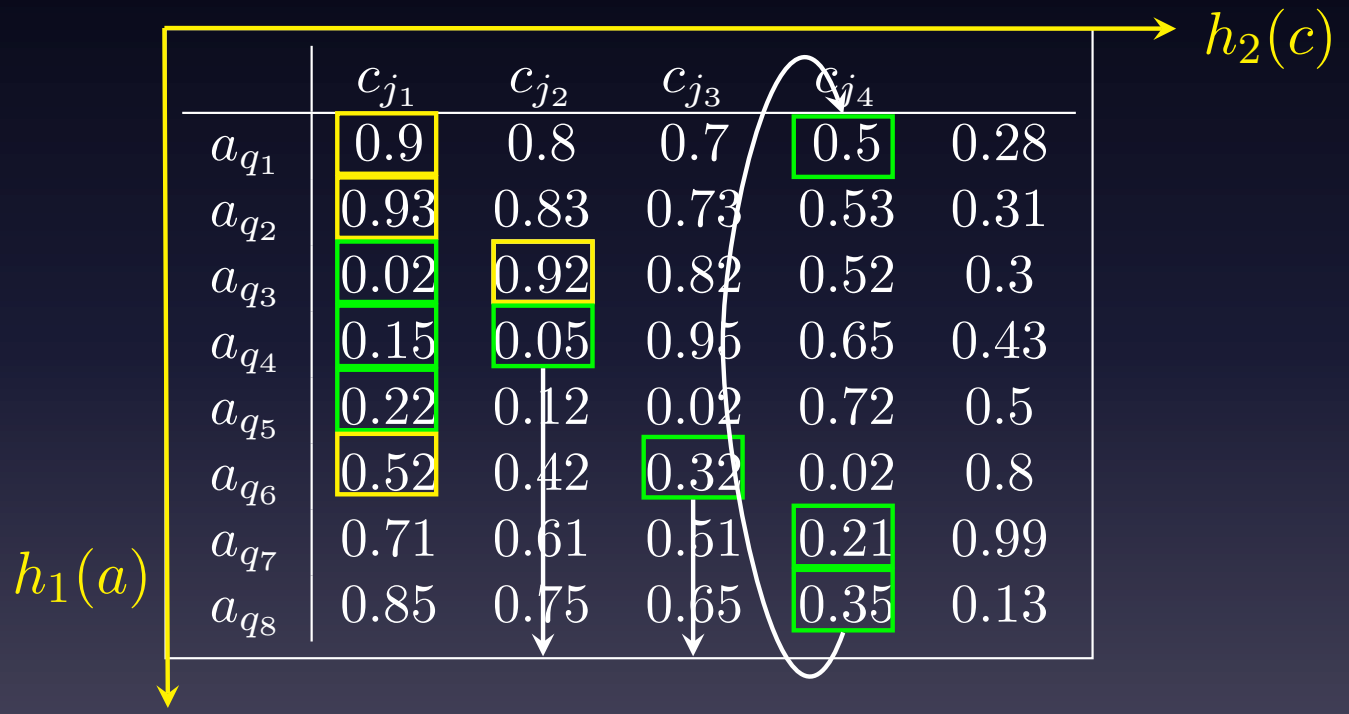

Time analysis: RAM

- Sorting the values of h_1 and h_2 takes $O(|A_i| + |C_i|)$ (sorting of pairwise independent values);
- given the threshold p, for each cluster we expect $O(p|A_i||C_i|)$ scanned pairs;

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$$A_i \times C_i$$

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- We choose p in order to have $O(\sum_i p|A_i||C_i|) = O(n)$ and enough reported pairs ($\geq k$).

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- Sorting the values of h_1 and h_2 takes $O(|A_i| + |C_i|)$ (sorting of pairwise independent values);
- given the threshold p, for each cluster we expect $O(p|A_i||C_i|)$ scanned pairs;

- merging the two buffers costs $O(k)$ but at least k pairs have to be scanned before, so $O(1)$ amortized;
- We choose p in order to have $O(\sum_i p|A_i||C_i|) = O(n)$ and enough reported pairs ($\geq k$).

Total cost: $O(n)$.

Time analysis: I/O

- Single global (cache oblivious) sorting of values $(b, h_1(a))$ and $(b, h_2(c))$;
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Total cost: $O(sort(n))$ I/Os.

Error probability

- Three possible sources of errors:
  - the kth value v found is too small $\rightarrow$ estimate too large;
  - the kth value v found is too large $\rightarrow$ estimate too small;
  - too few values are found smaller than $p \rightarrow$ we allow an additive error.
- analysis essentially equivalent in all the cases;
- follows the one in Bar-Yossef et al.;
- error probability for all of the cases: 1/3.

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  - in expected linear time;
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If you would be a real seeker after truth,
you must at least once in your life doubt,
as far as possible,
all things.