Generating k-independent variables in constant time

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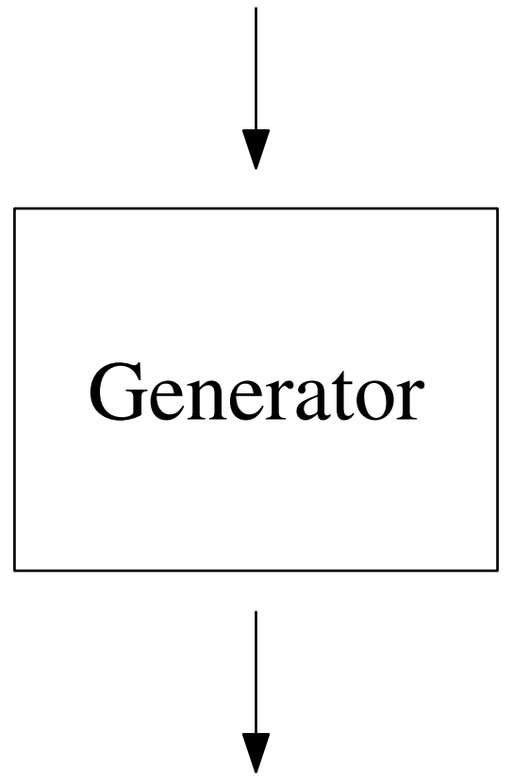
Random seed

11010100 01101101



Pseudorandom output sequence

00011011 11111100 10010101 11010001 11100110

k-independent variables

Generator output:

- Sequence of k-independent variables over a finite field $\mathbb{F}$

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- k-independence:
  - Y_i uniformly distributed over $\mathbb{F}_p$
  - The variables at every set of k distinct positions are independent

$$\Pr[Y_{i_1} = y_1, Y_{i_2} = y_2, \ldots, Y_{i_k} = y_k] = p^{-k}$$

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- Many properties of full randomness for $k \ll p$

Model and problem

Model:

- Word-RAM model
  - Elements of $\mathbb{F}_p$ fit in a single word
  - Constant time addition and multiplication in $\mathbb{F}_p$

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Problem:

- Construct a data structure that outputs a k-independent sequence over $\mathbb{F}_p$, one variable at the time
  - Low time complexity of generating next element
  - Space usage and initialization time close to k

Main result

Explicit data structure for generating the next element of a k-independent sequence in worst case *constant time* using space and initialization time $k \operatorname{poly} \log k$

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COMPARISON WITH PREVIOUS RESULTS

Reference	**Space**	**Generation time**
Multipoint evaluation [GG'13]	$O(k \log k)$	$O(\log^2 k \log \log k)$
Expander hashing [Siegel'89]	$O(p^\varepsilon),\ \varepsilon > 0$	$O(1)$
Our generator	$O(k \operatorname{poly} \log k)$	$O(1)$

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Polynomial hash functions [Joffe'74, CW'81]

- Choose k random elements from $\mathbb{F}_p$ and use them as coefficients of a polynomial

$$h(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{k-1} x^{k-1} \bmod p$$

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- Polynomial hash function as a generator
  - Space k to store coefficients
  - Evaluation time $O(k)$

$h(0)$	$h(1)$	$h(2)$	$\cdot \quad \cdot \quad \cdot$	$h(p-1)$

- Space usage near-optimal with respect to sequence length

k-independent hashing: lower bound

- What is the best time-space tradeoff we can hope for using hashing?

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- Cell probe lower bound [Siegel'89]

$$k\text{-independent hash function } h : \mathbb{F}_p \to \mathbb{F}_p$$

$$\text{Time } t < k \Rightarrow$$
$$\text{Space at least } p^{1/t} \text{ words}$$

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- Constructions in the word-RAM model still far from lower bound

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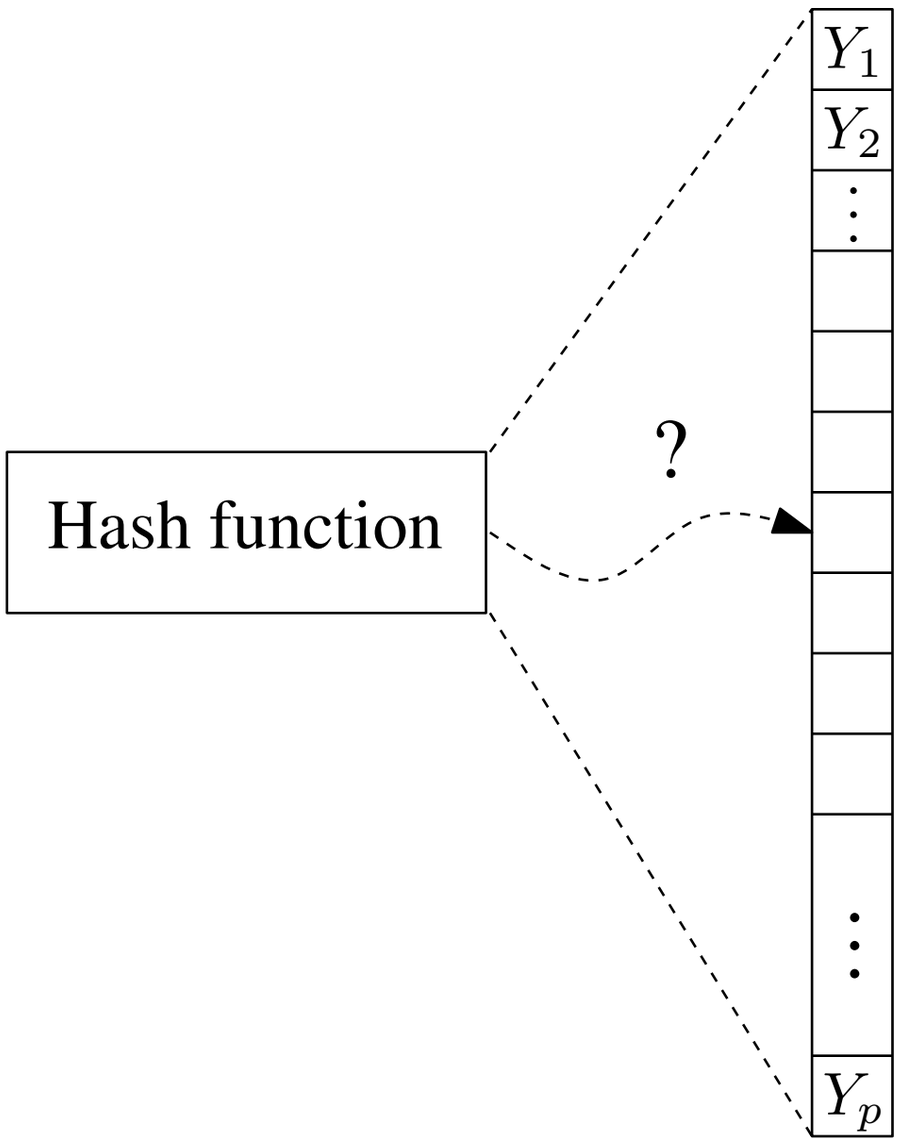

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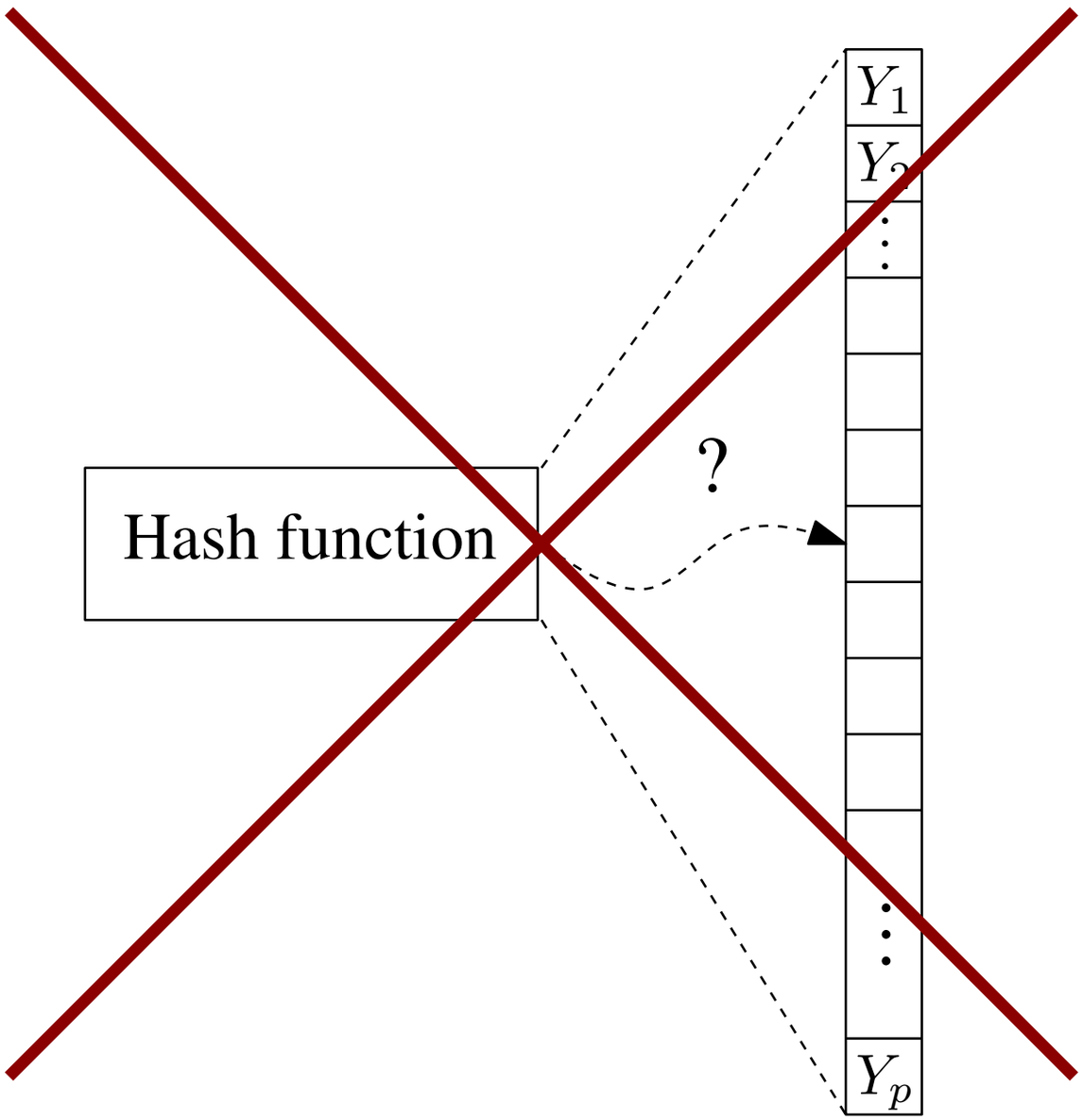

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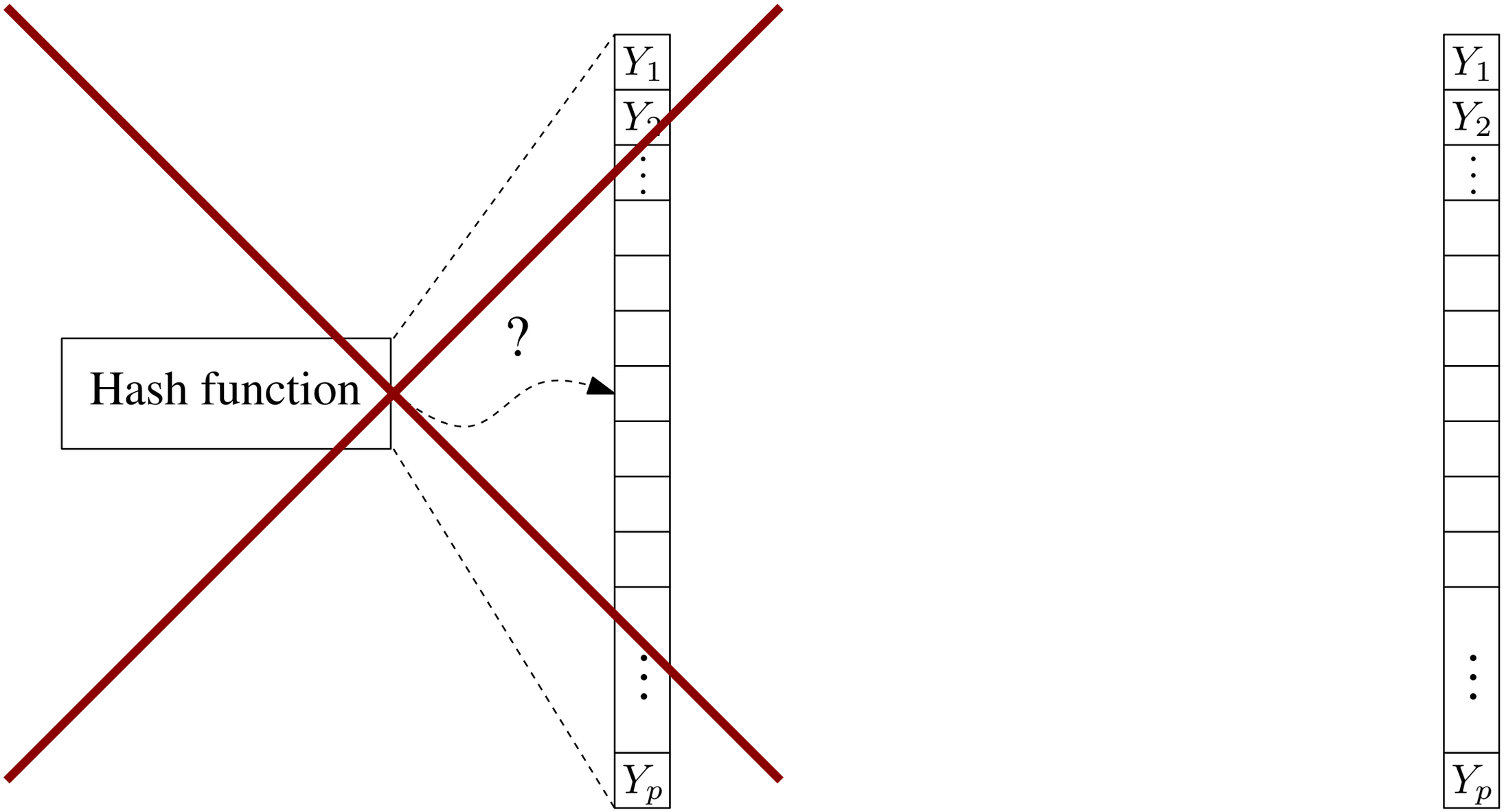

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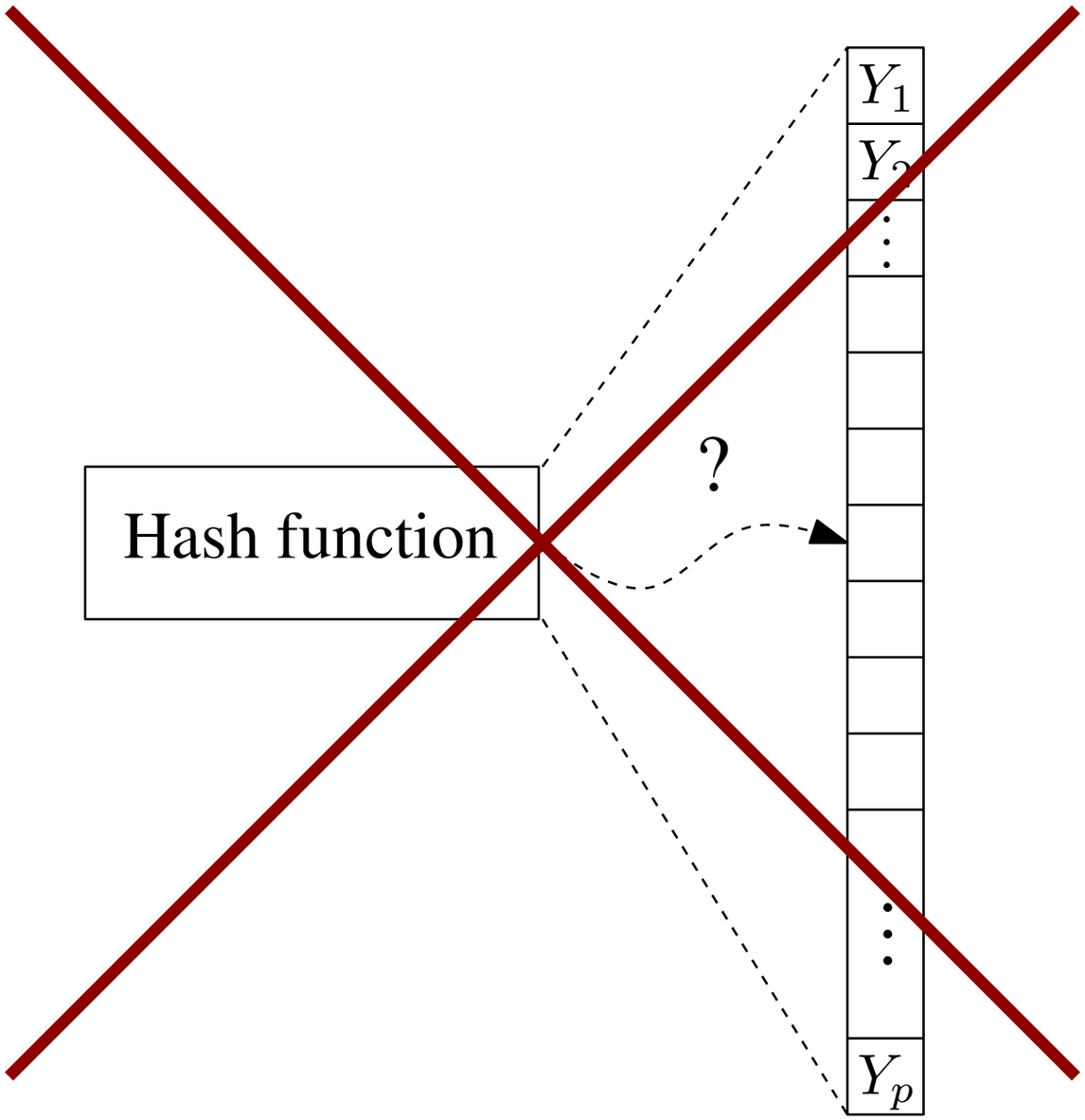

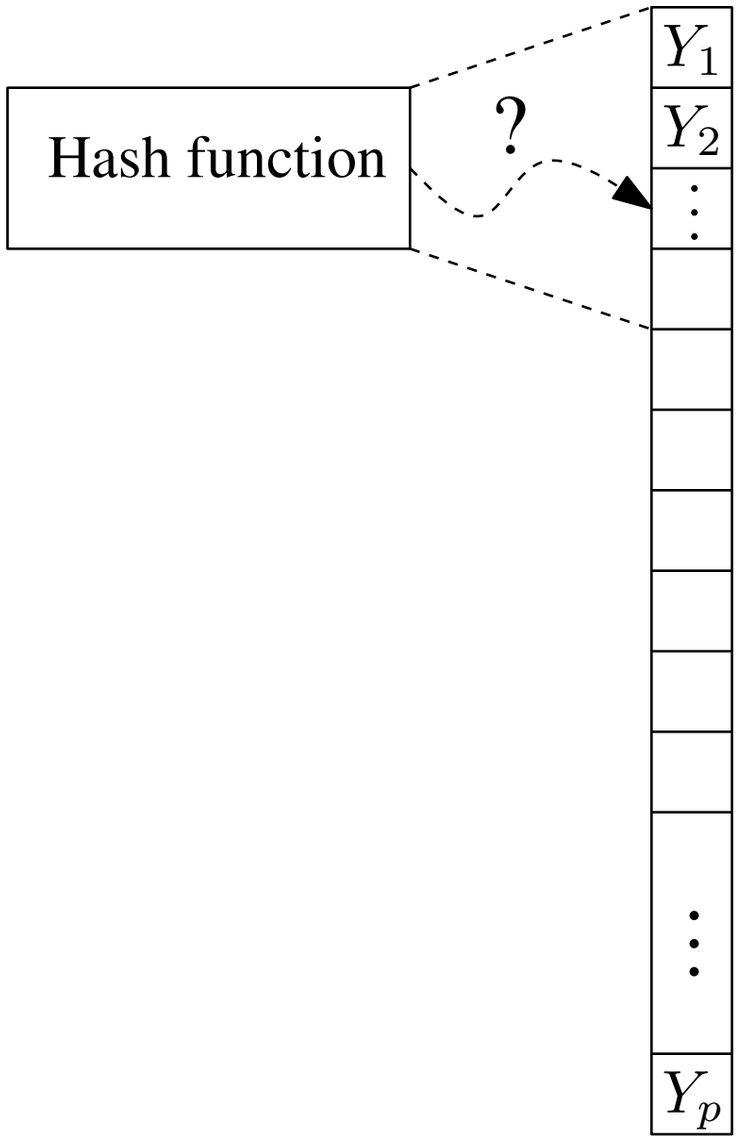

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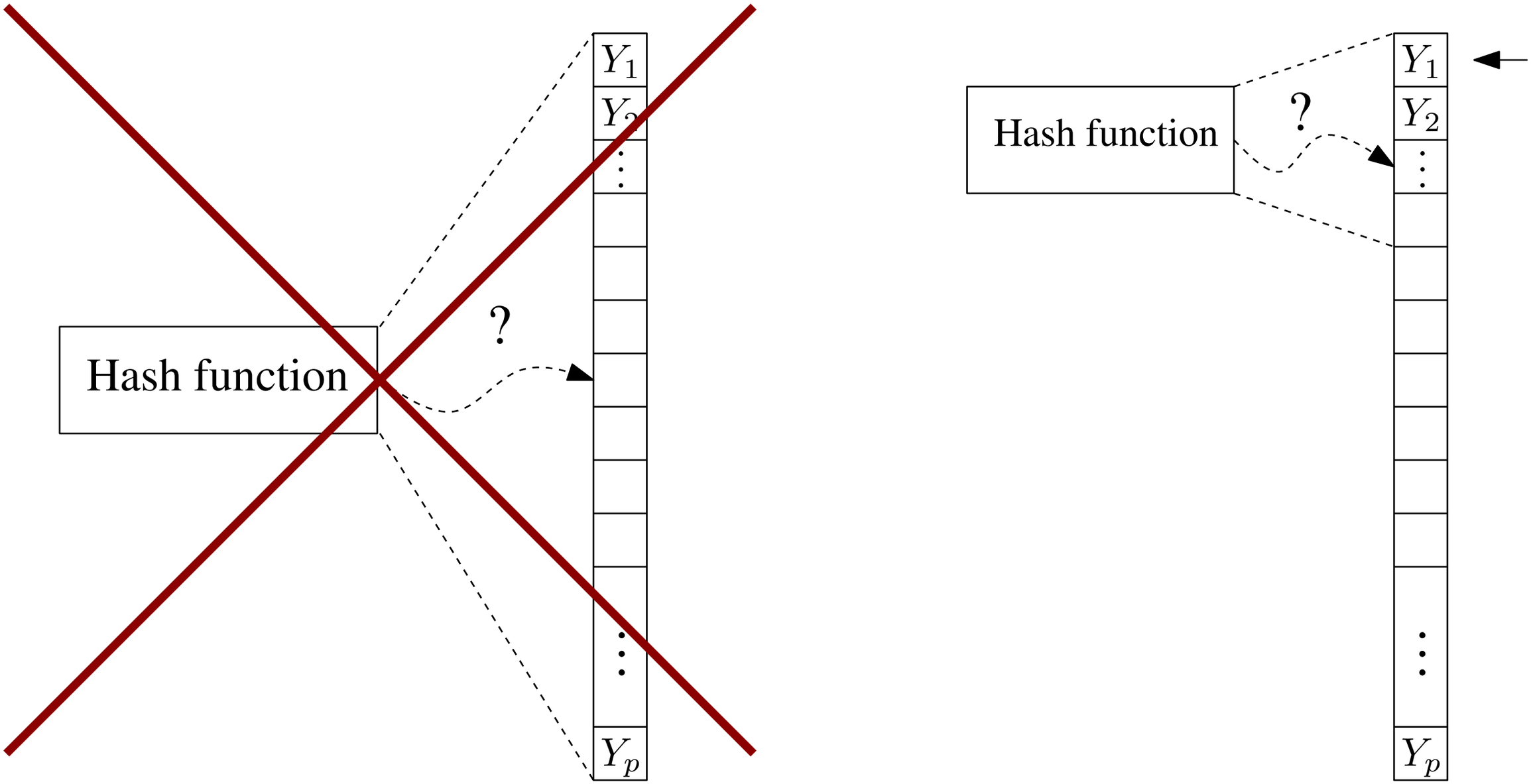

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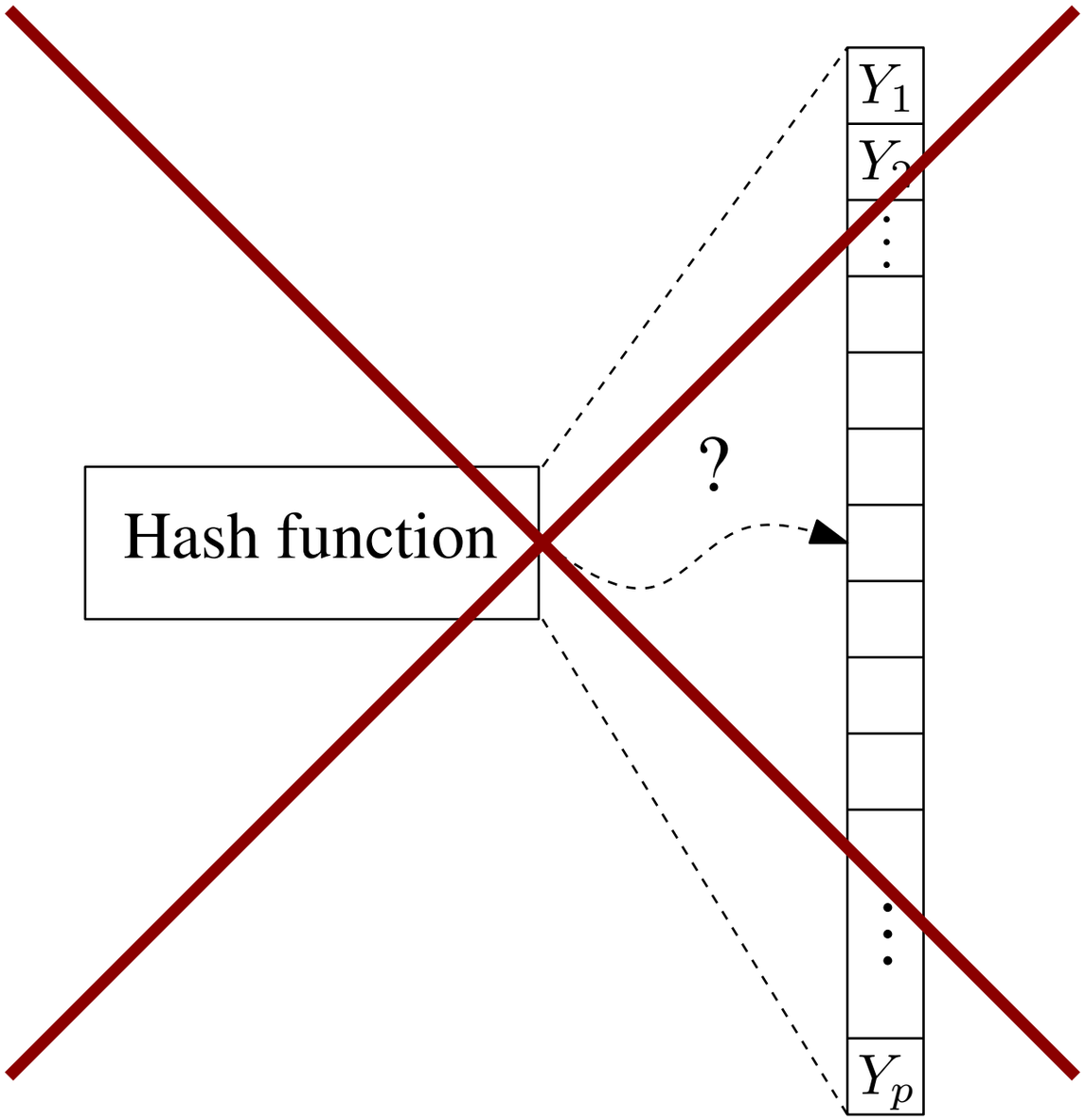

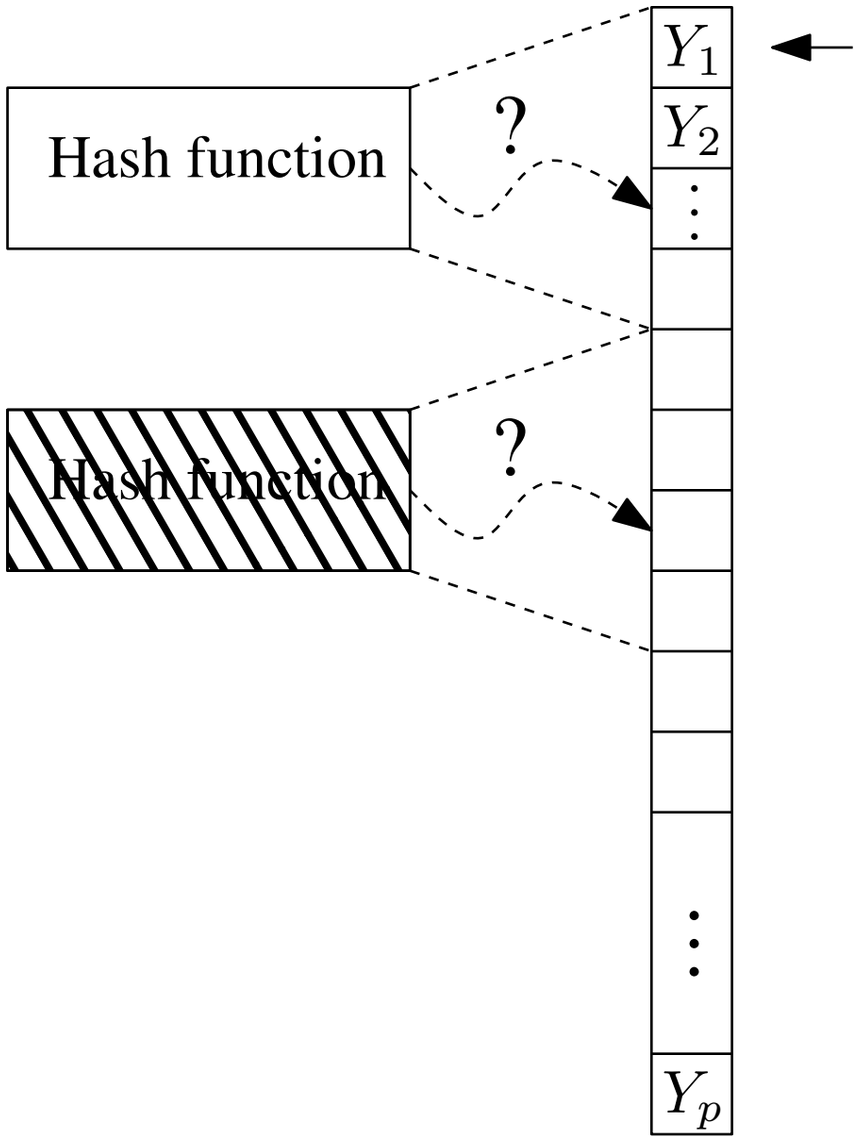

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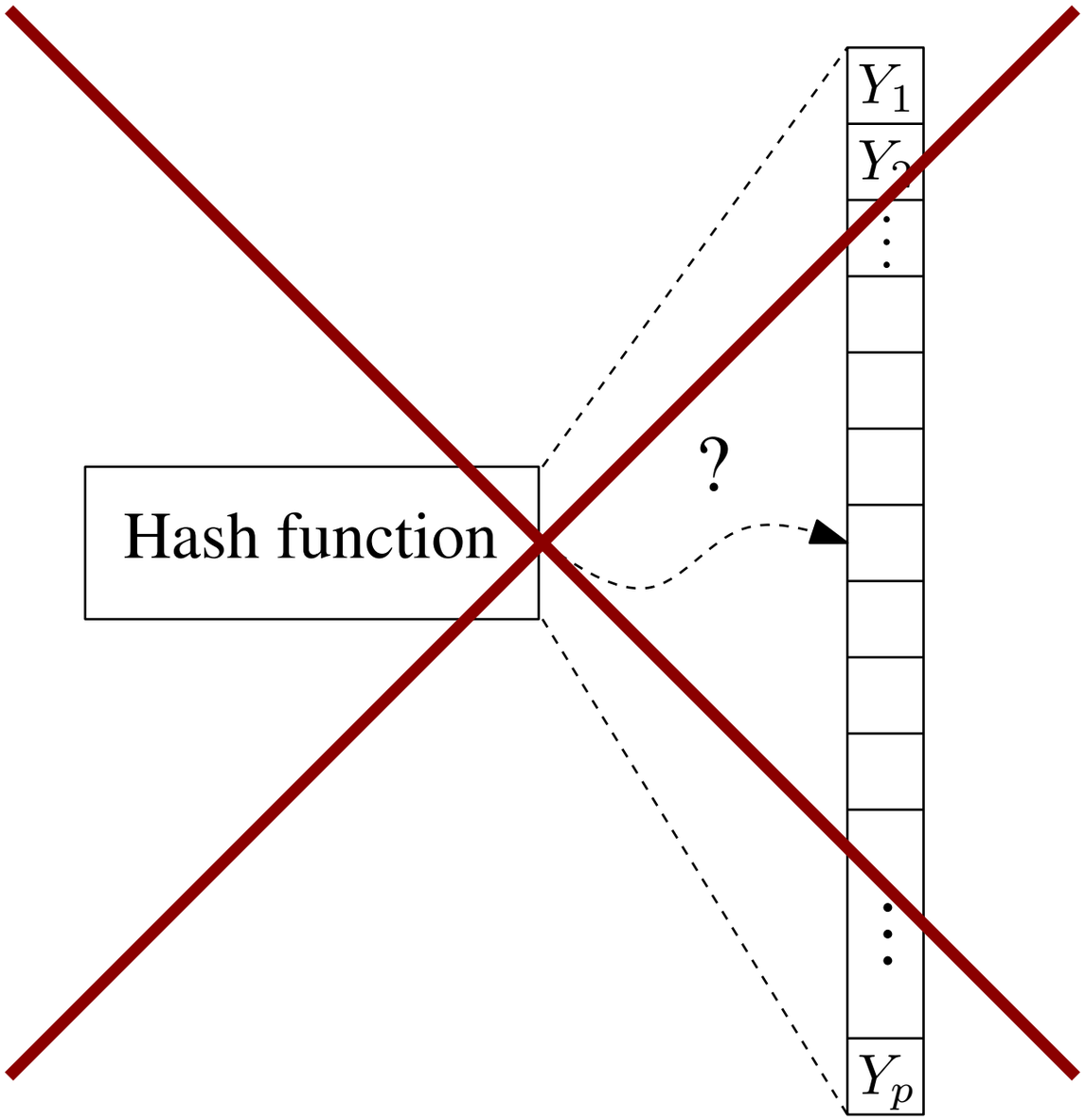

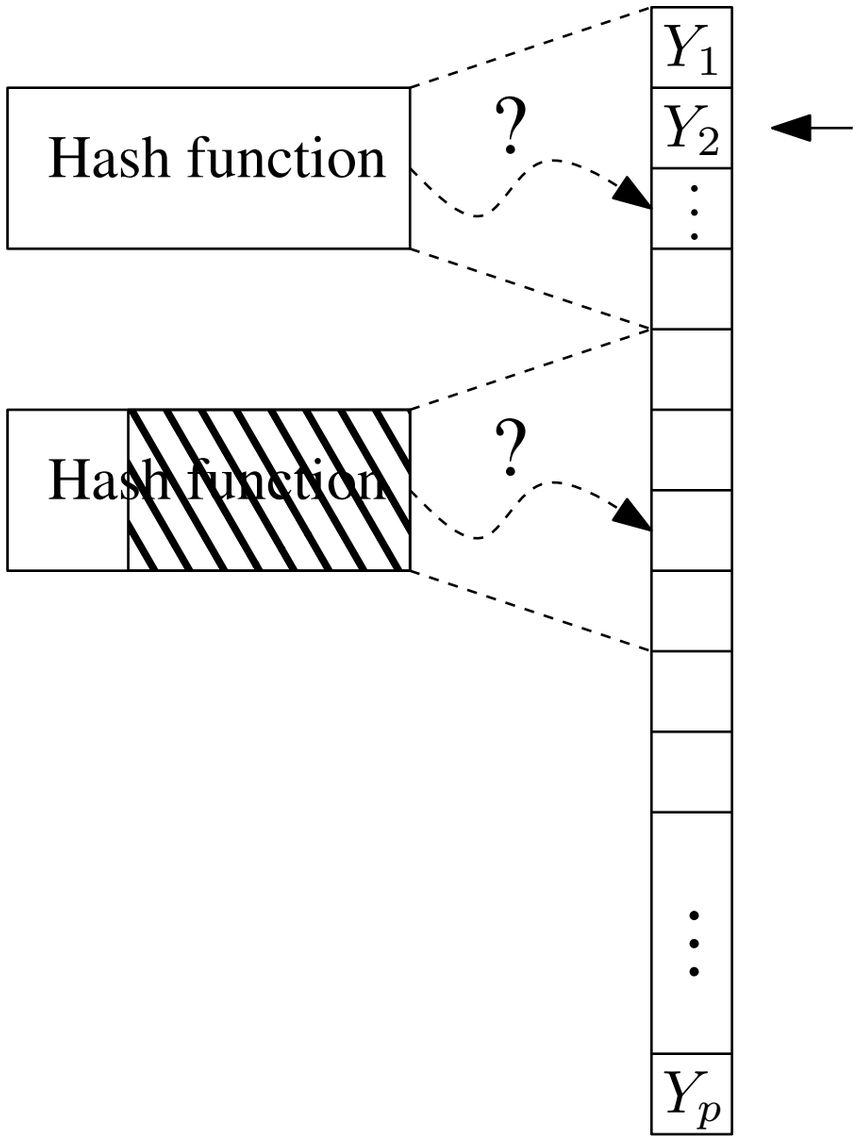

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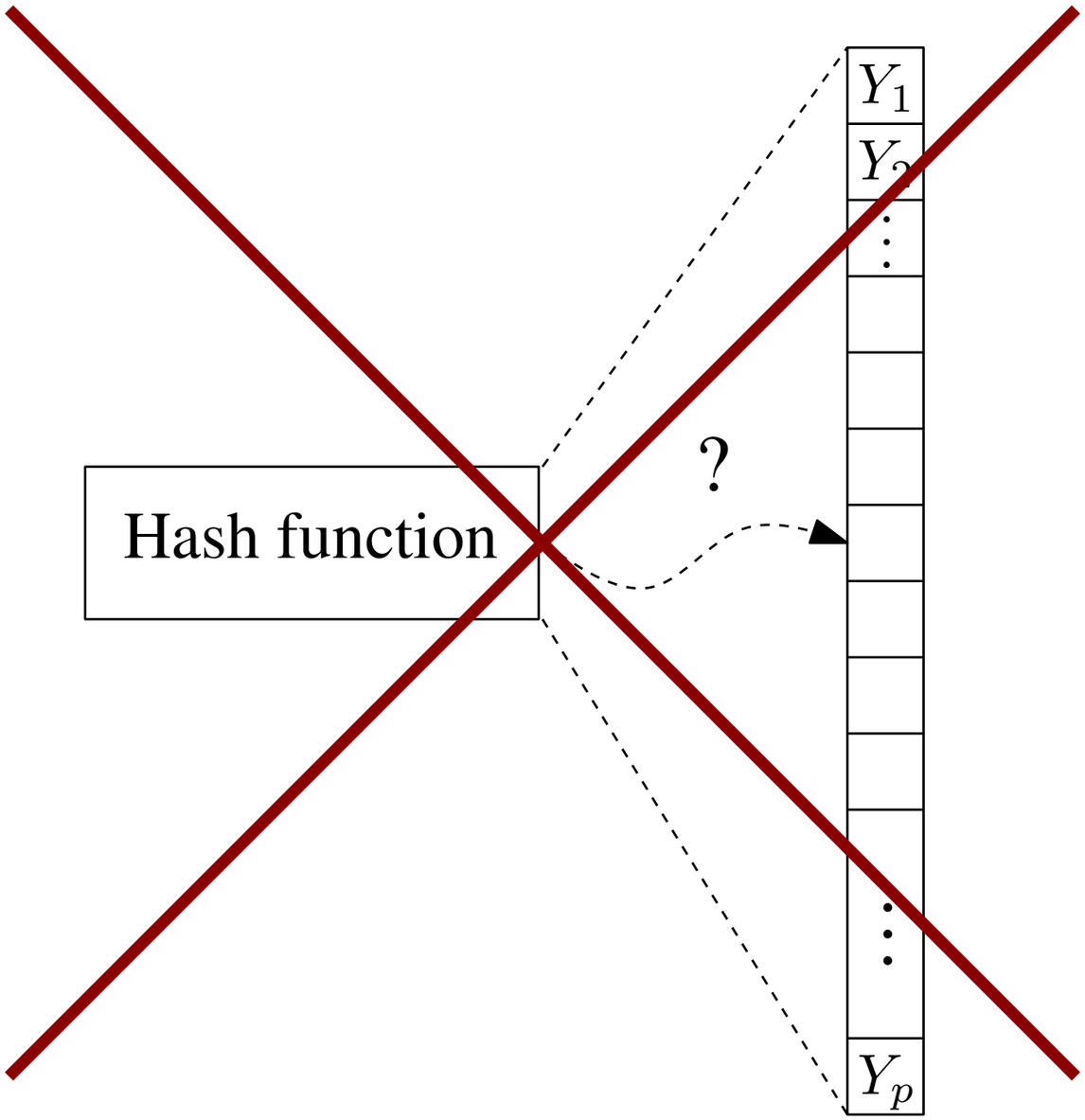

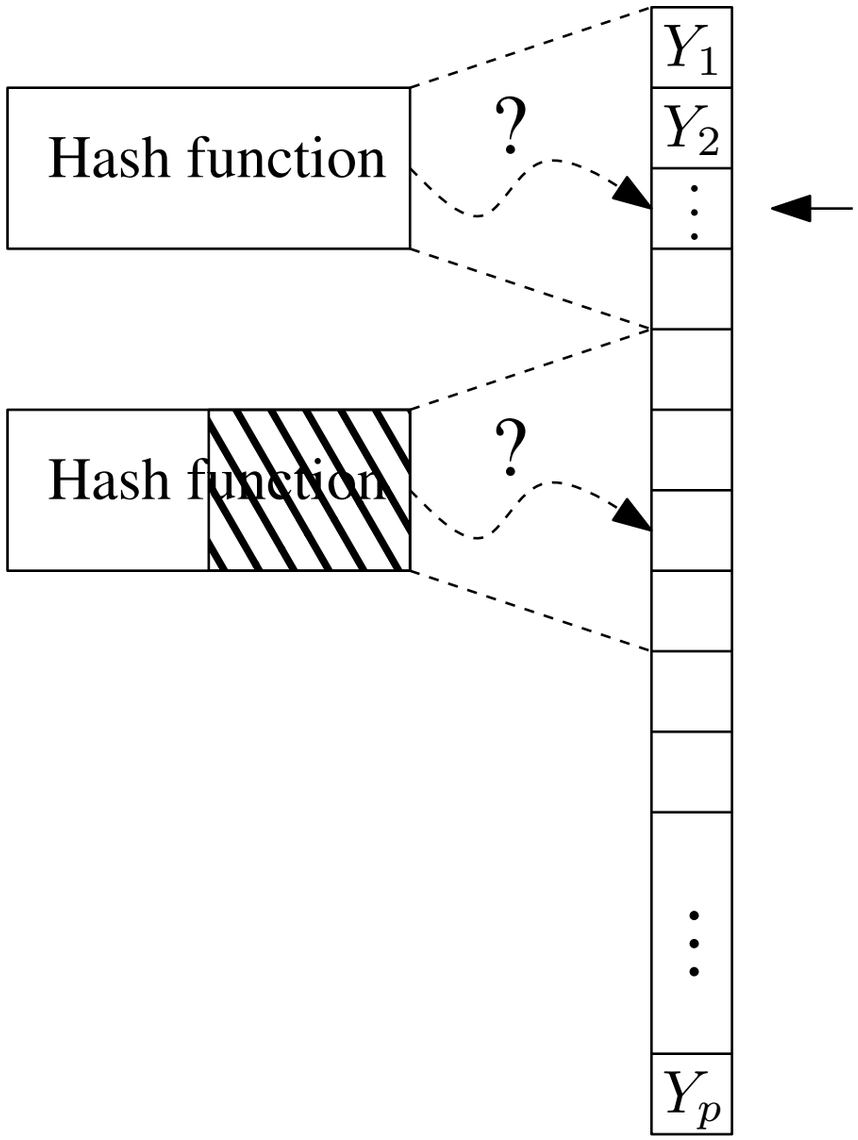

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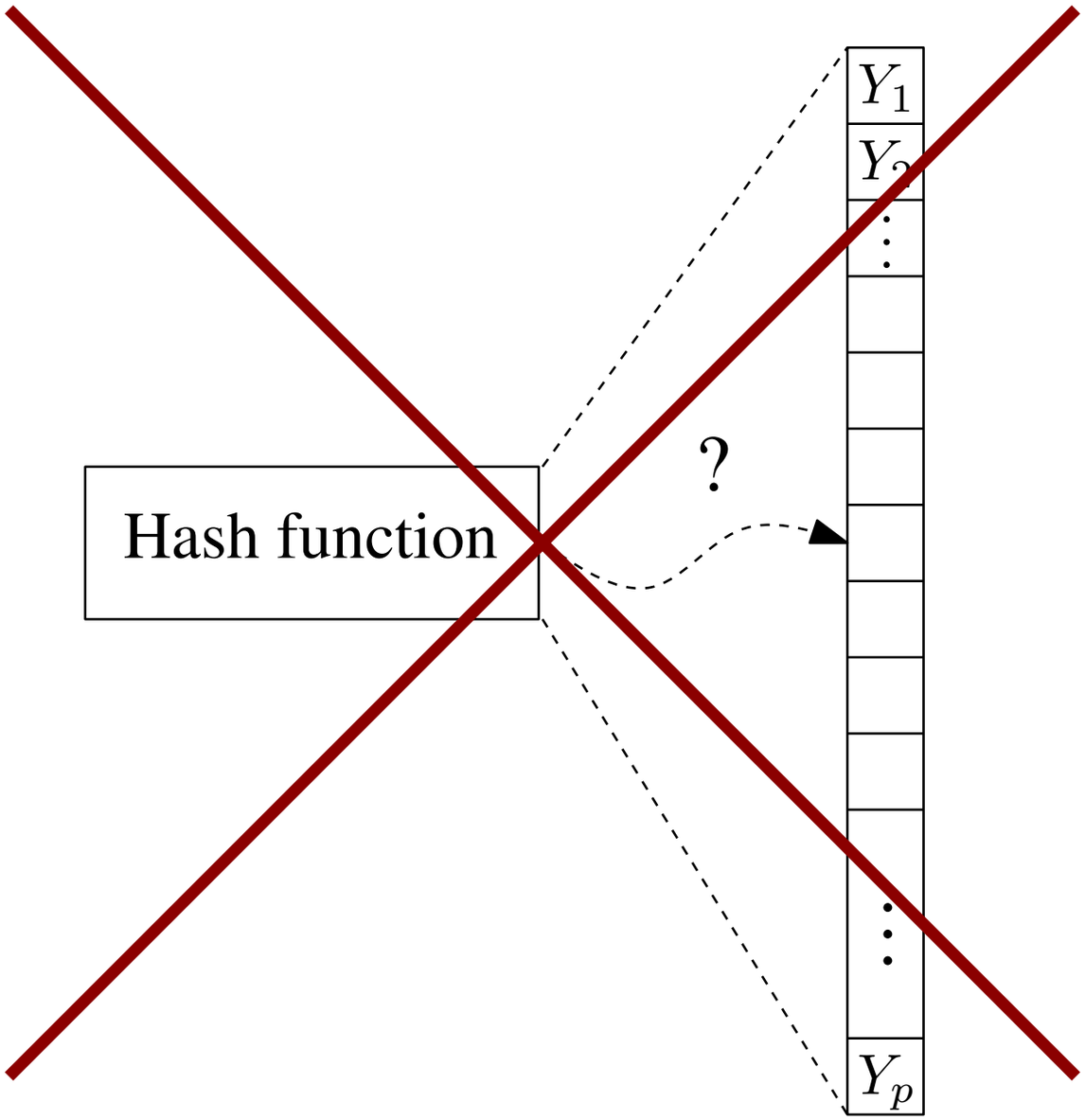

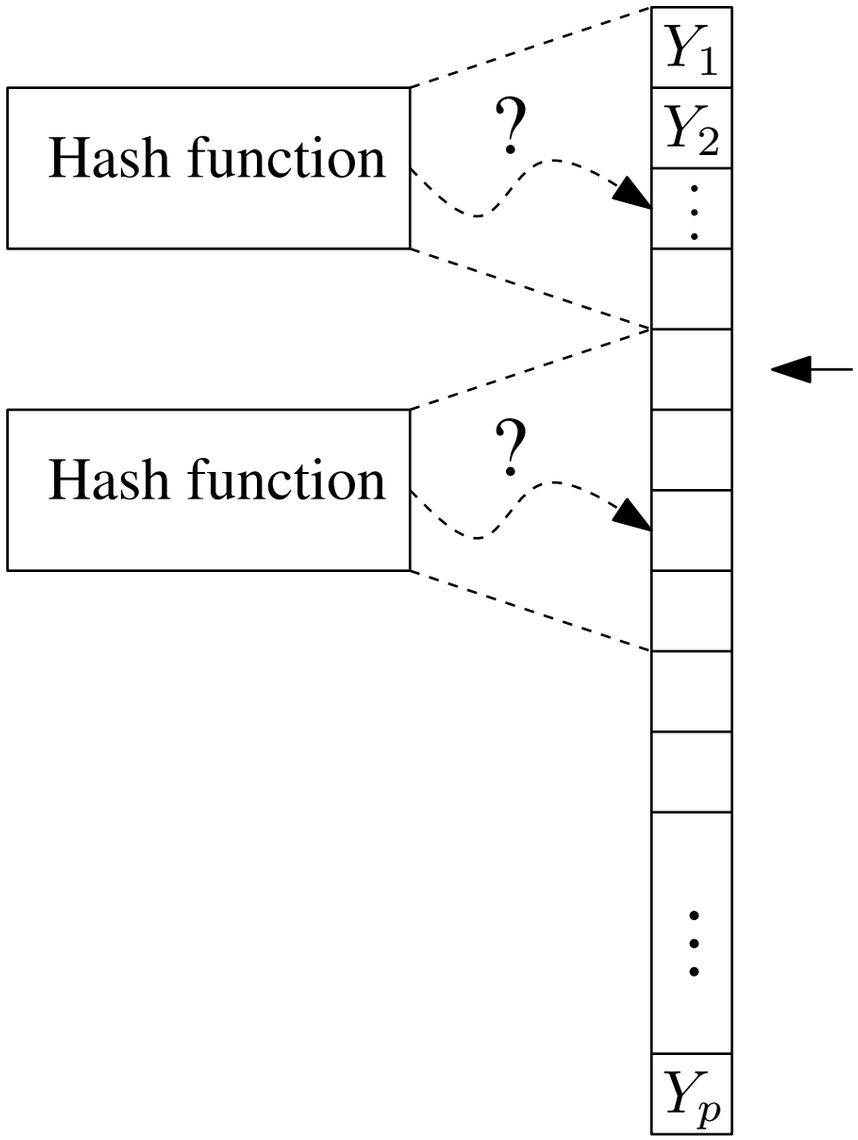

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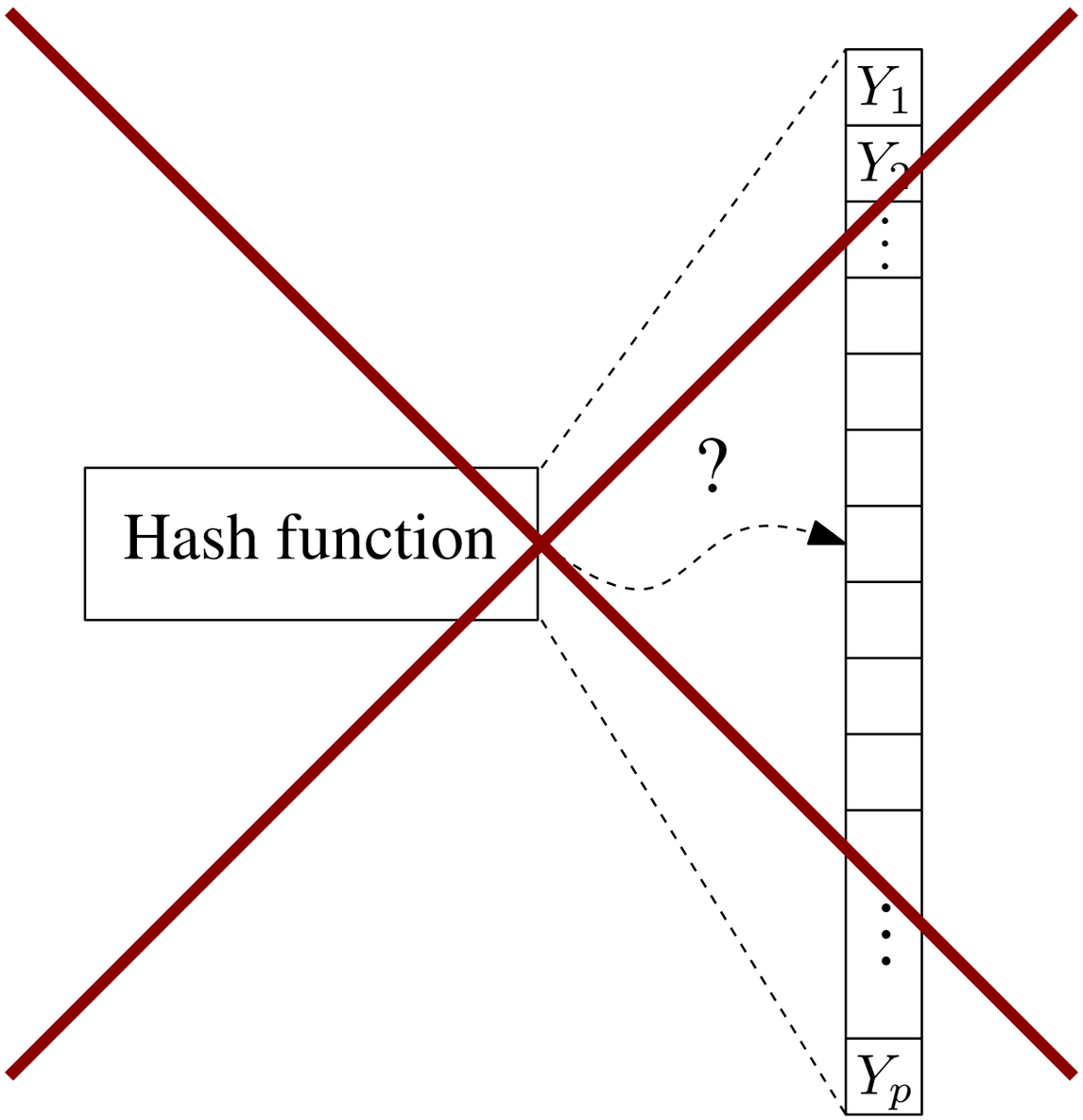

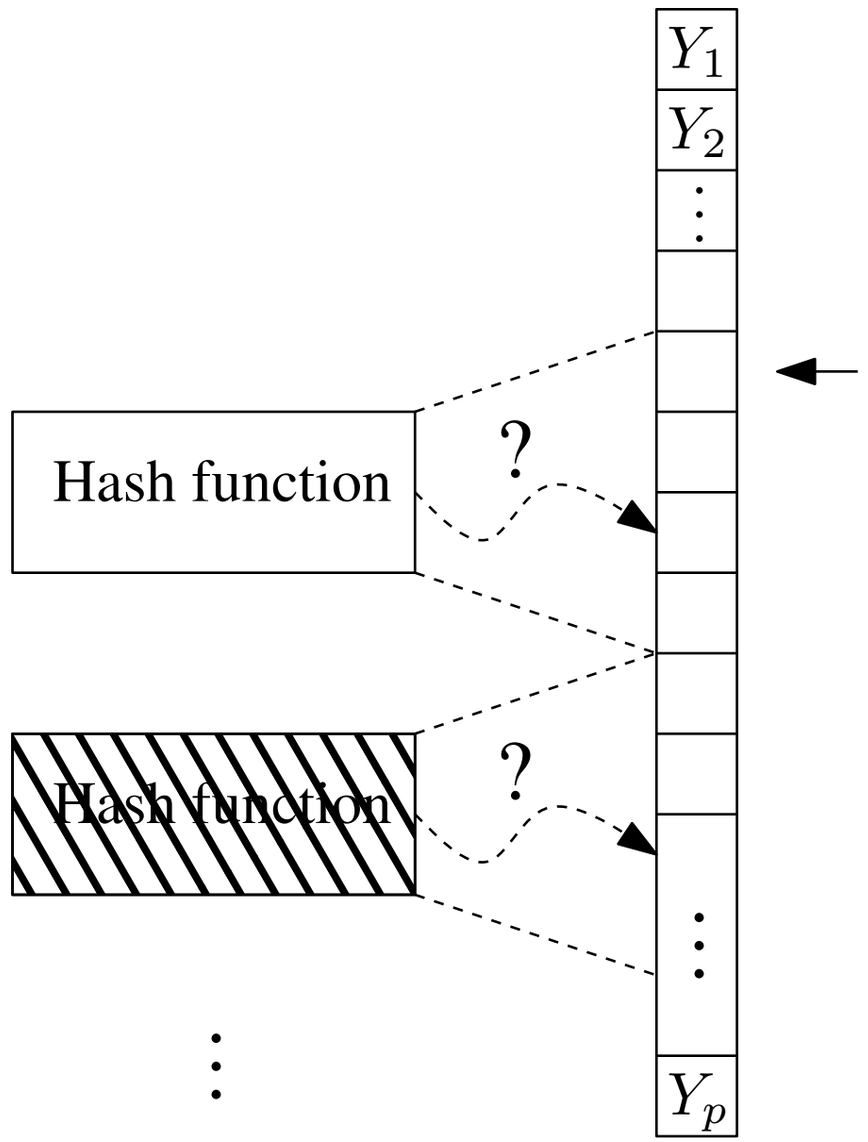

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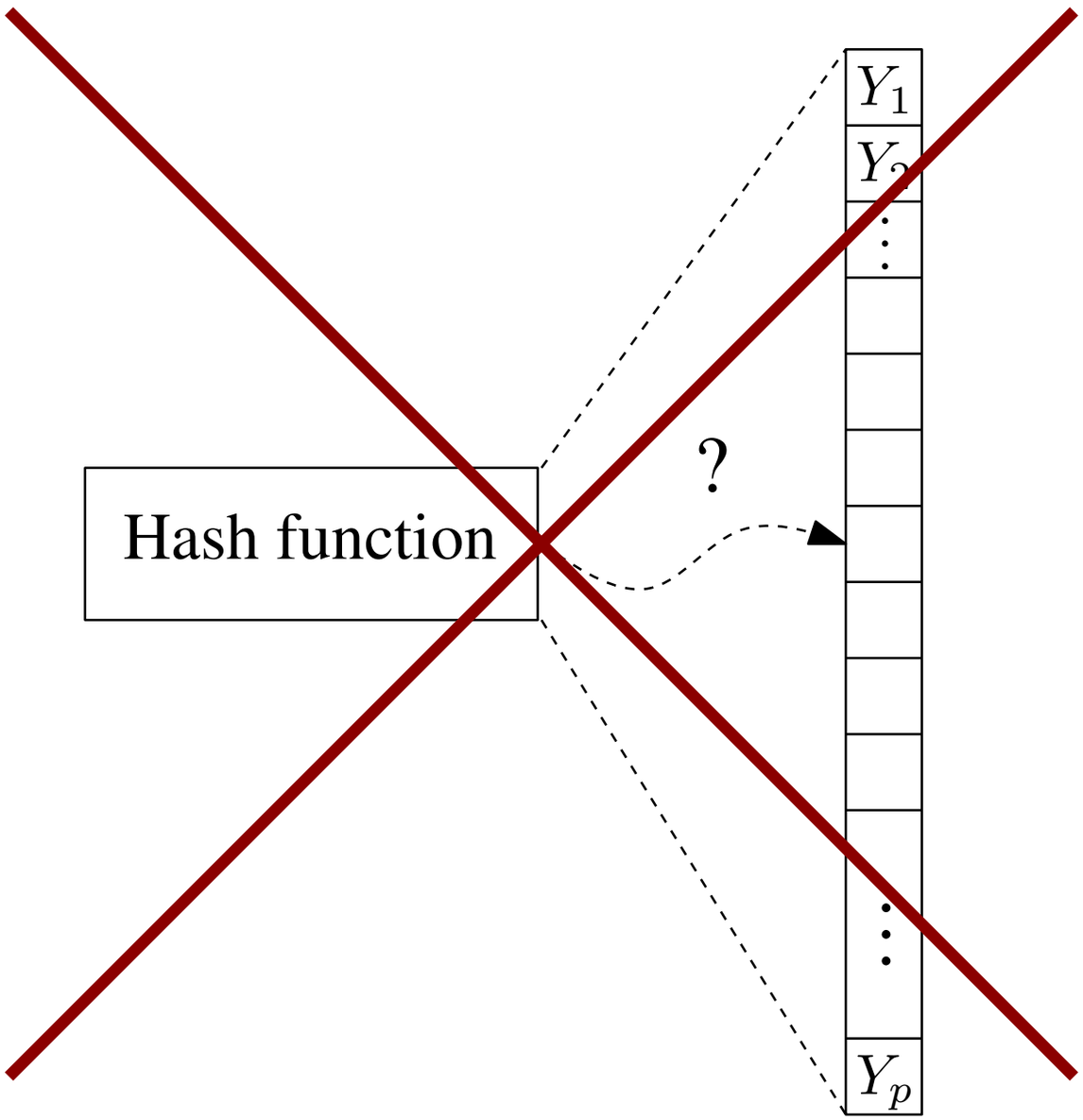

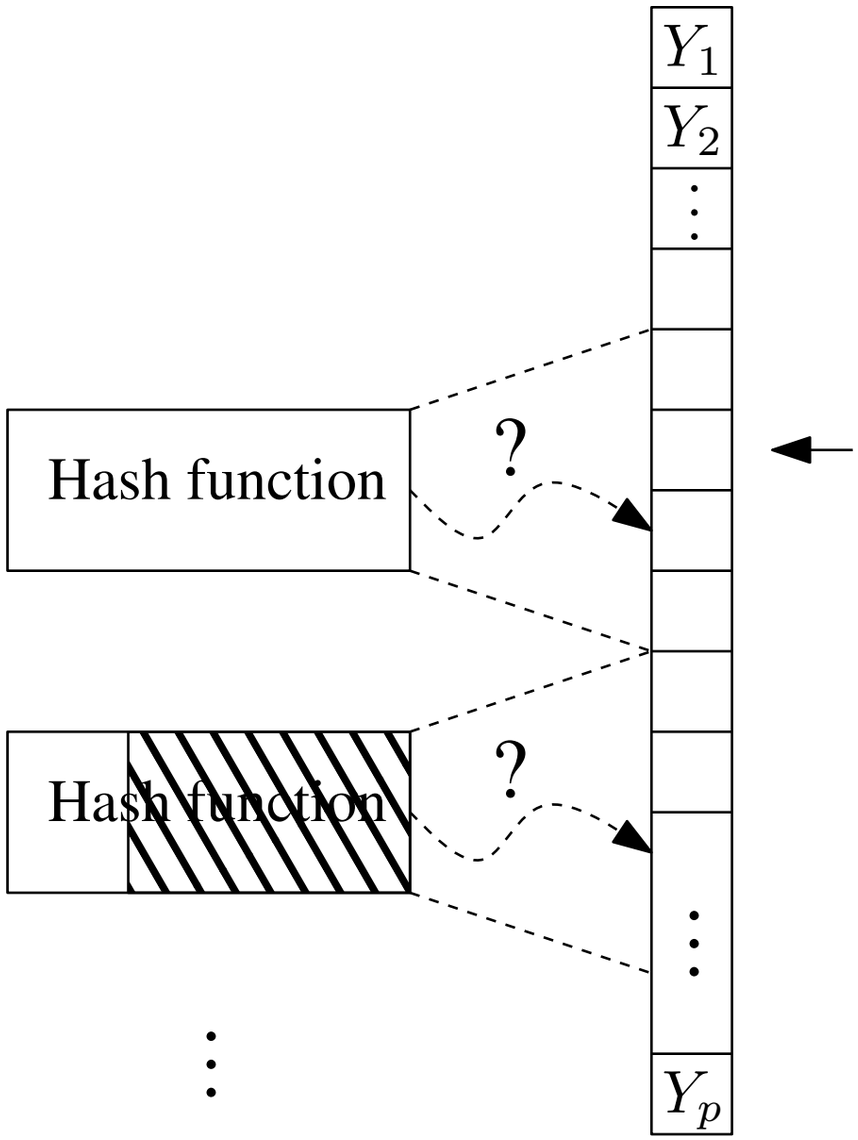

Construction overview

Initialization and evaluation of local hash functions:

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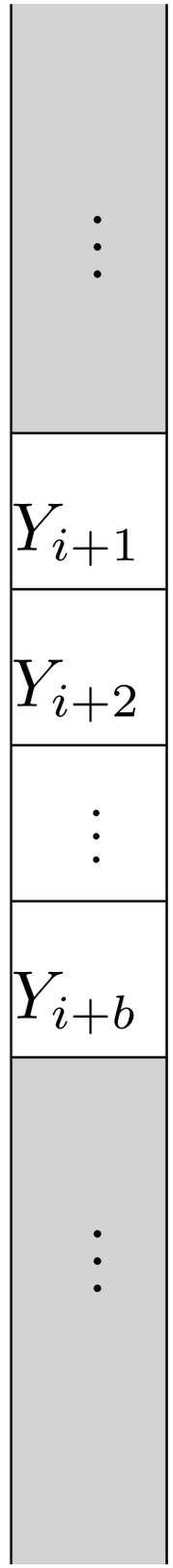
  - Evaluate in k points using time $k \operatorname{poly} \log k$
  - $\operatorname{poly} \log k$ time per value generated



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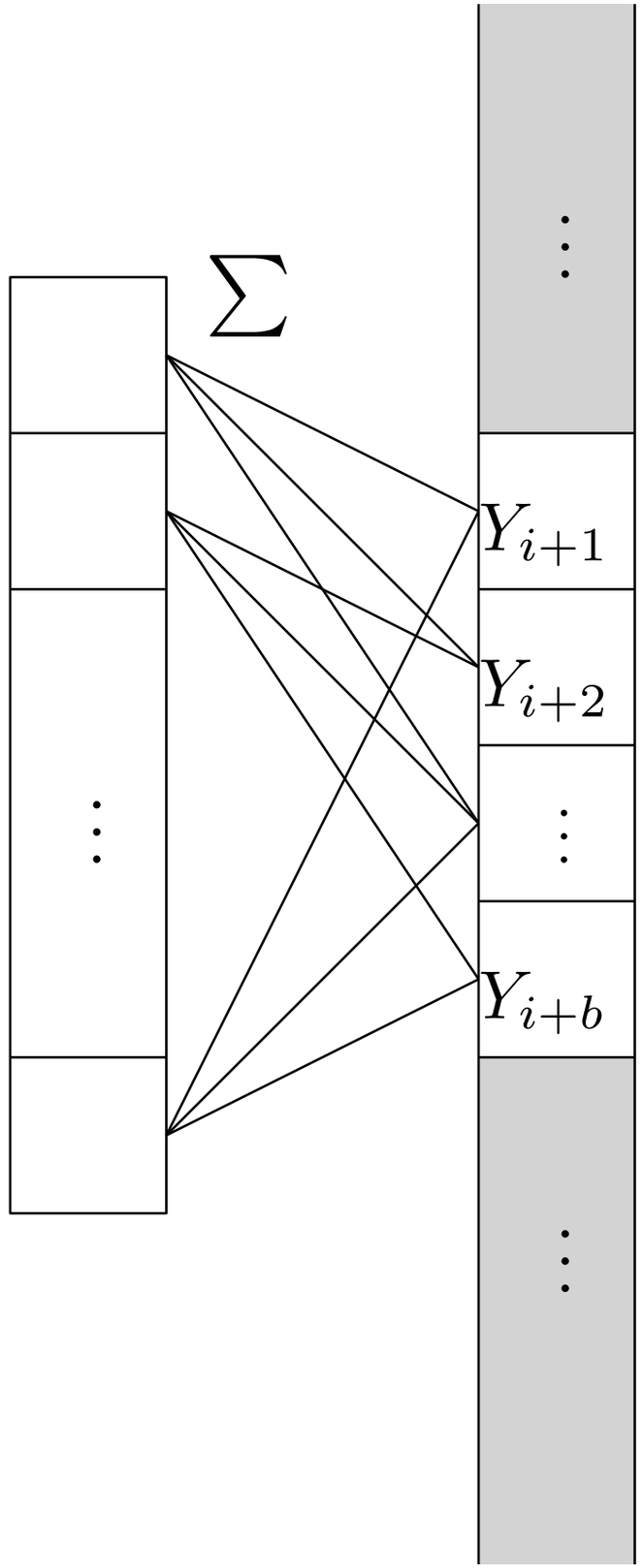
2. Expander hashing
   - Unbalanced bipartite expander, degree $O(1)$
     - Generate $k \operatorname{poly} \log k$ values from k values
     - Output is $\Omega(k)$-independent



Multipoint evaluation

- Assume that $\mathbb{F}_p$ contains a primitive kth root of unity ω_k

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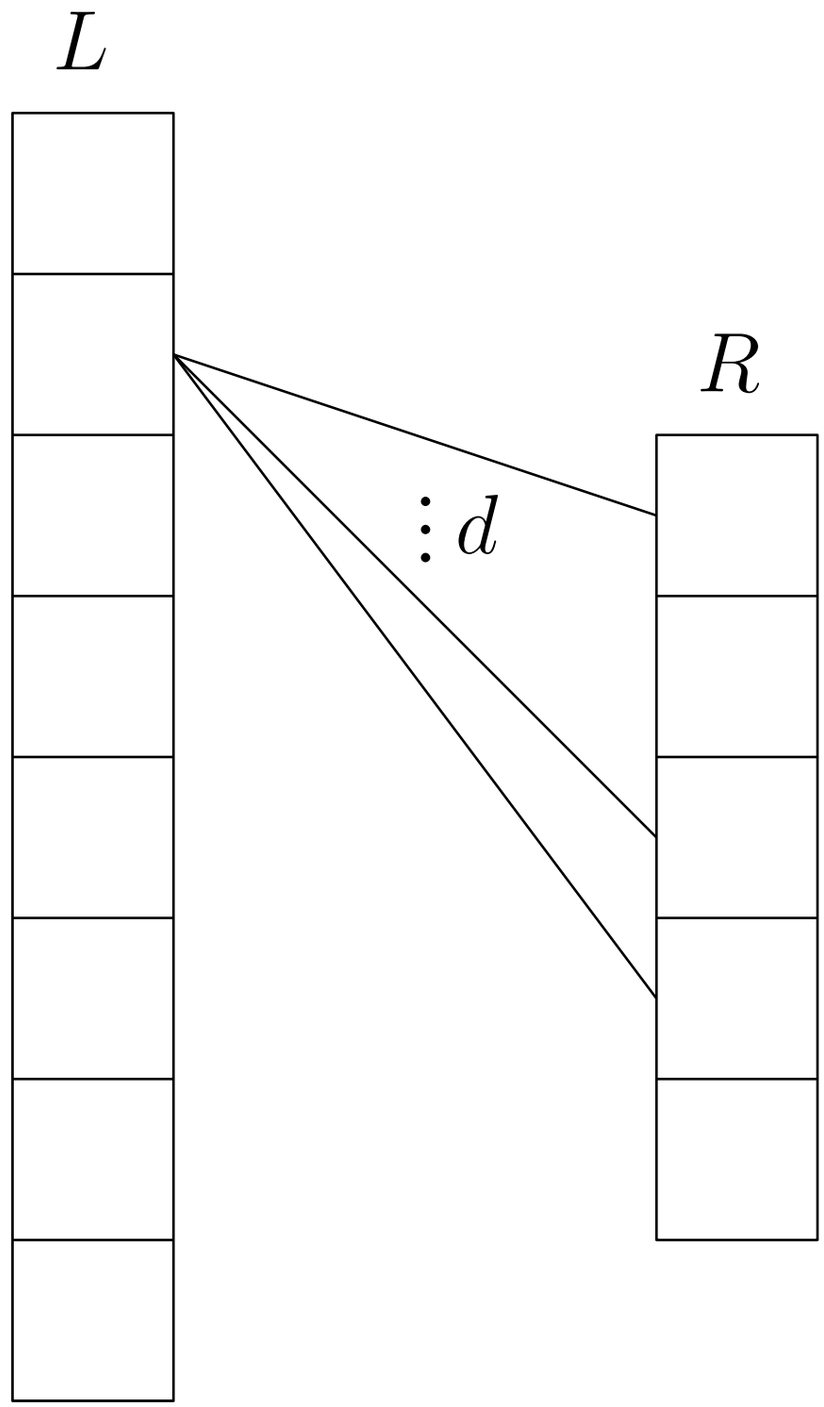
- General multipoint evaluation: Time $O(k \log^3 k)$ [GG'13]

Expander hashing

- Left d-regular bipartite graph Γ



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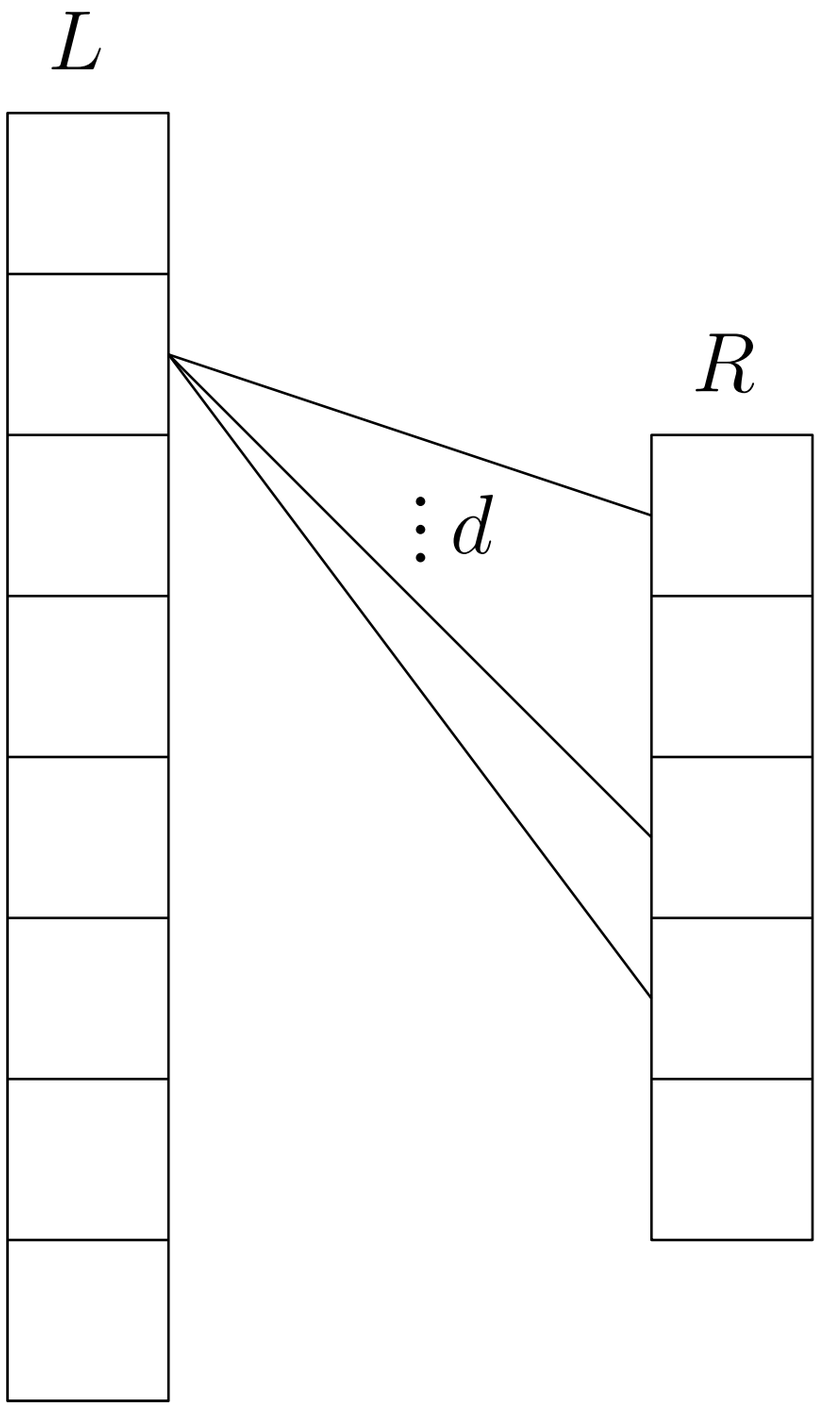

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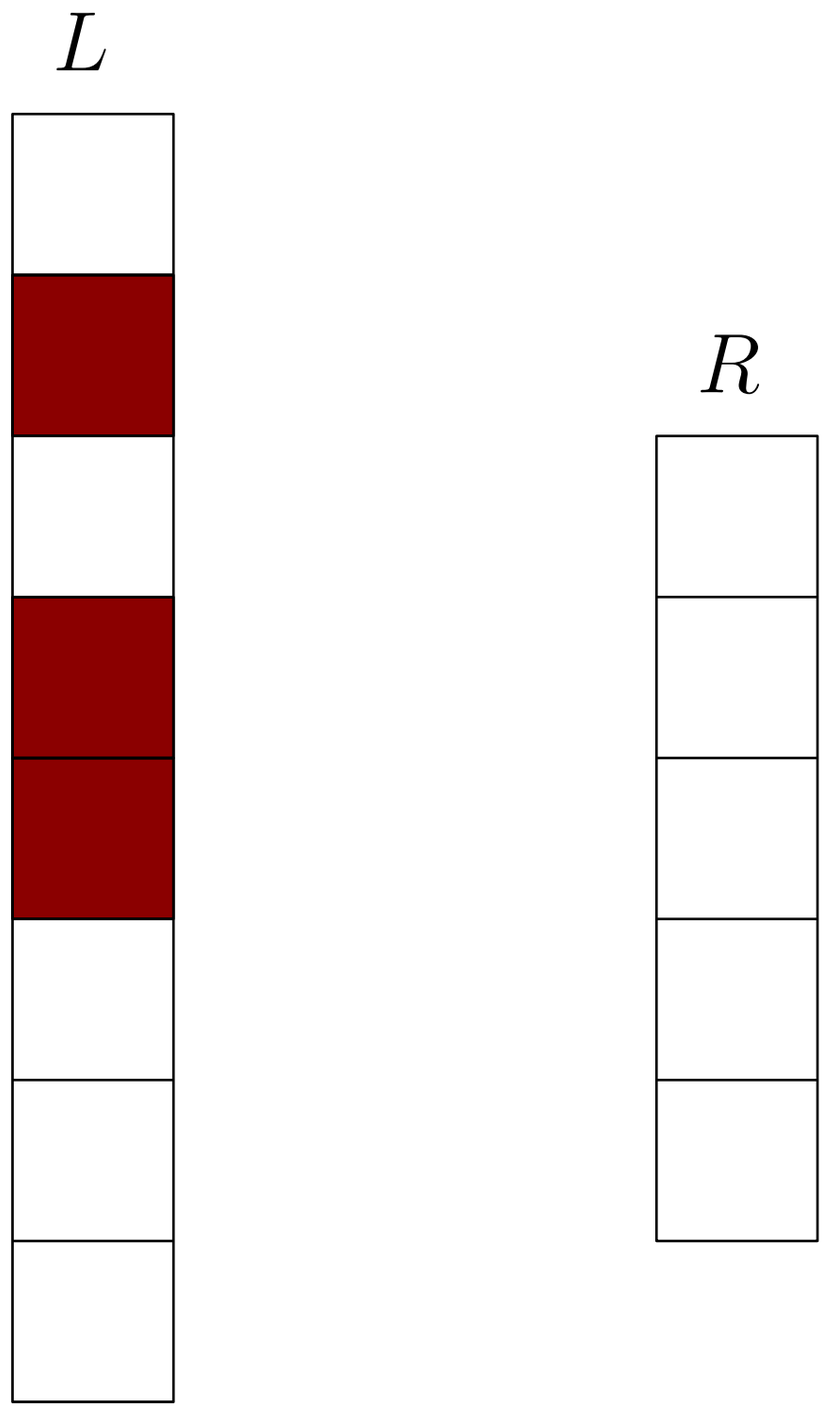

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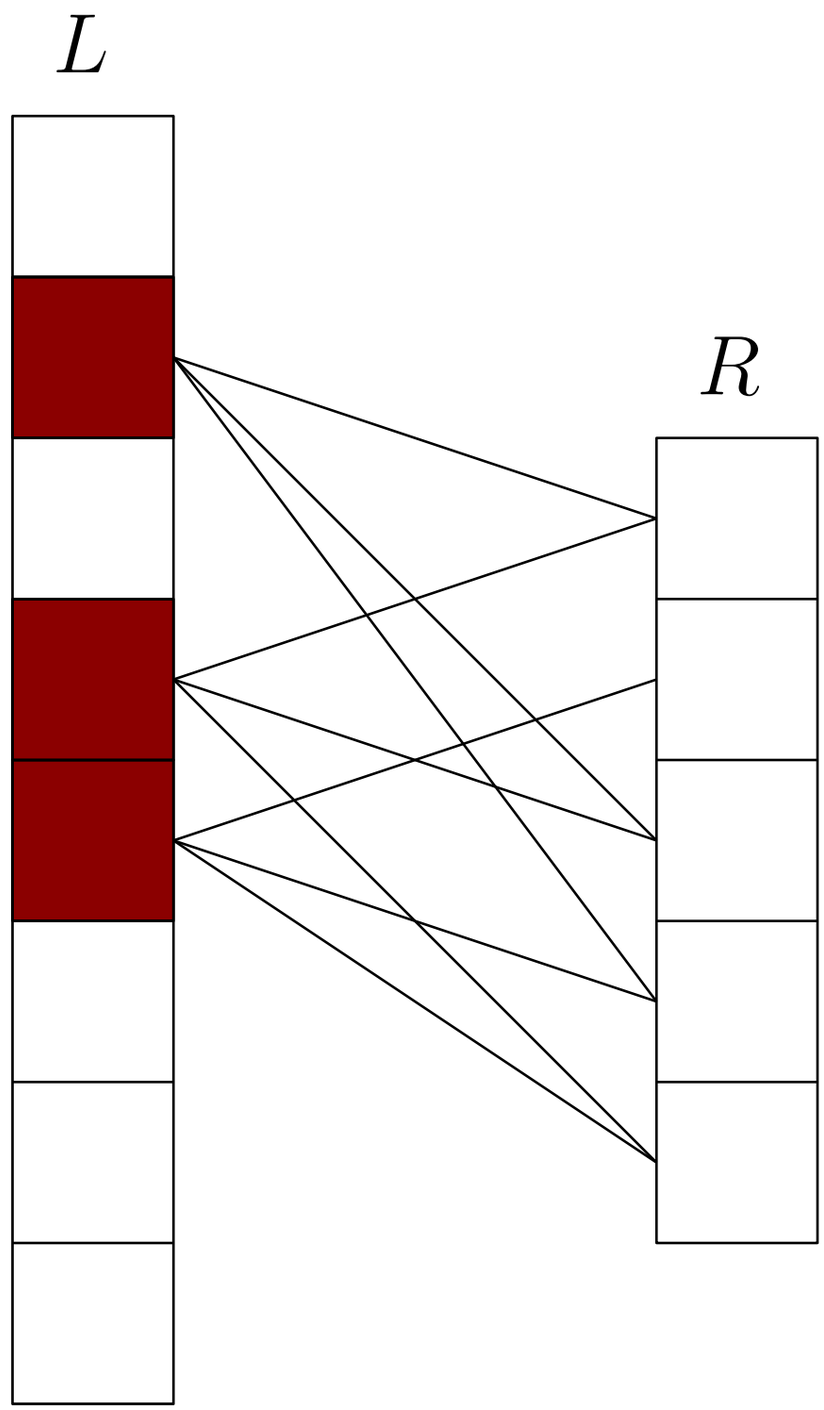

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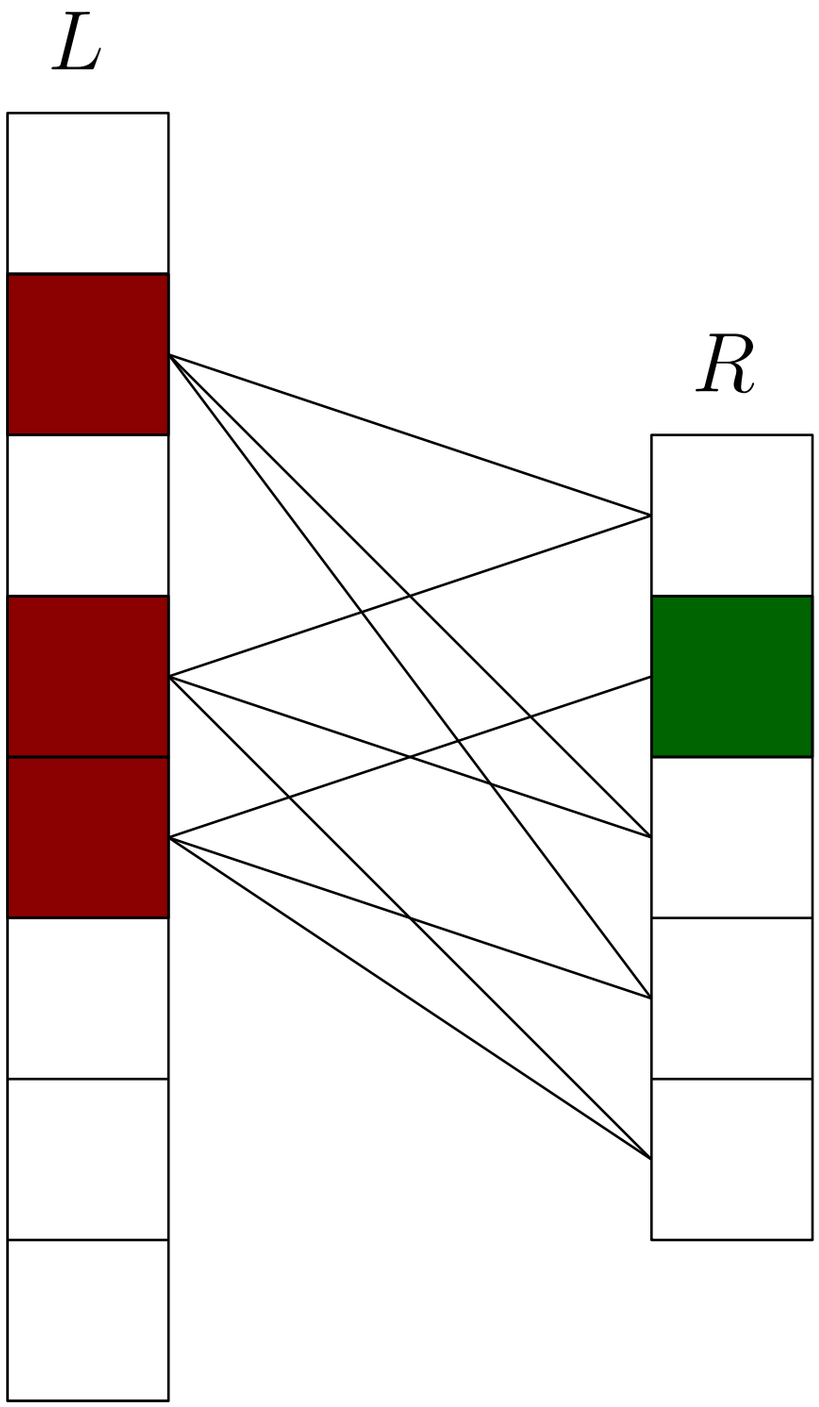

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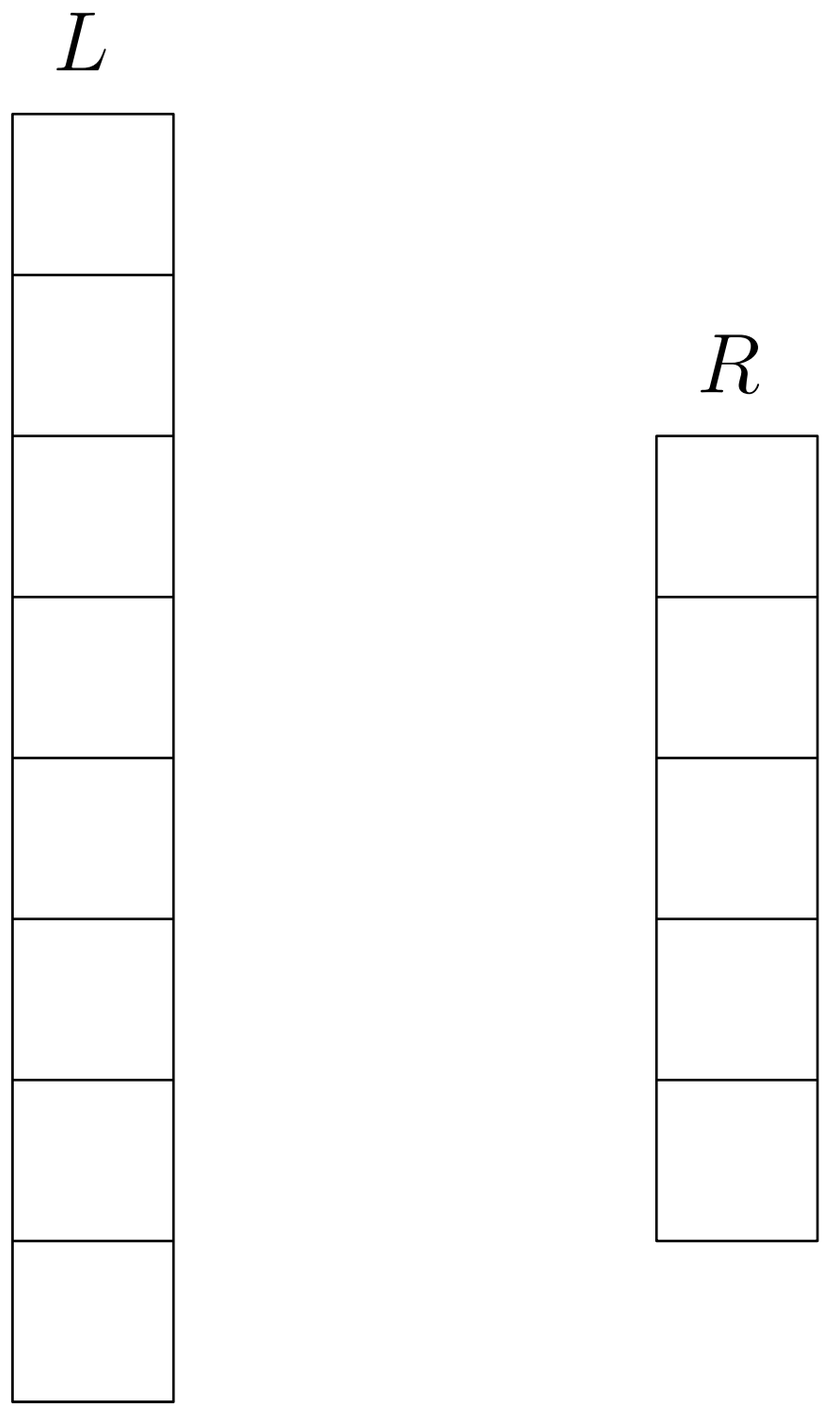

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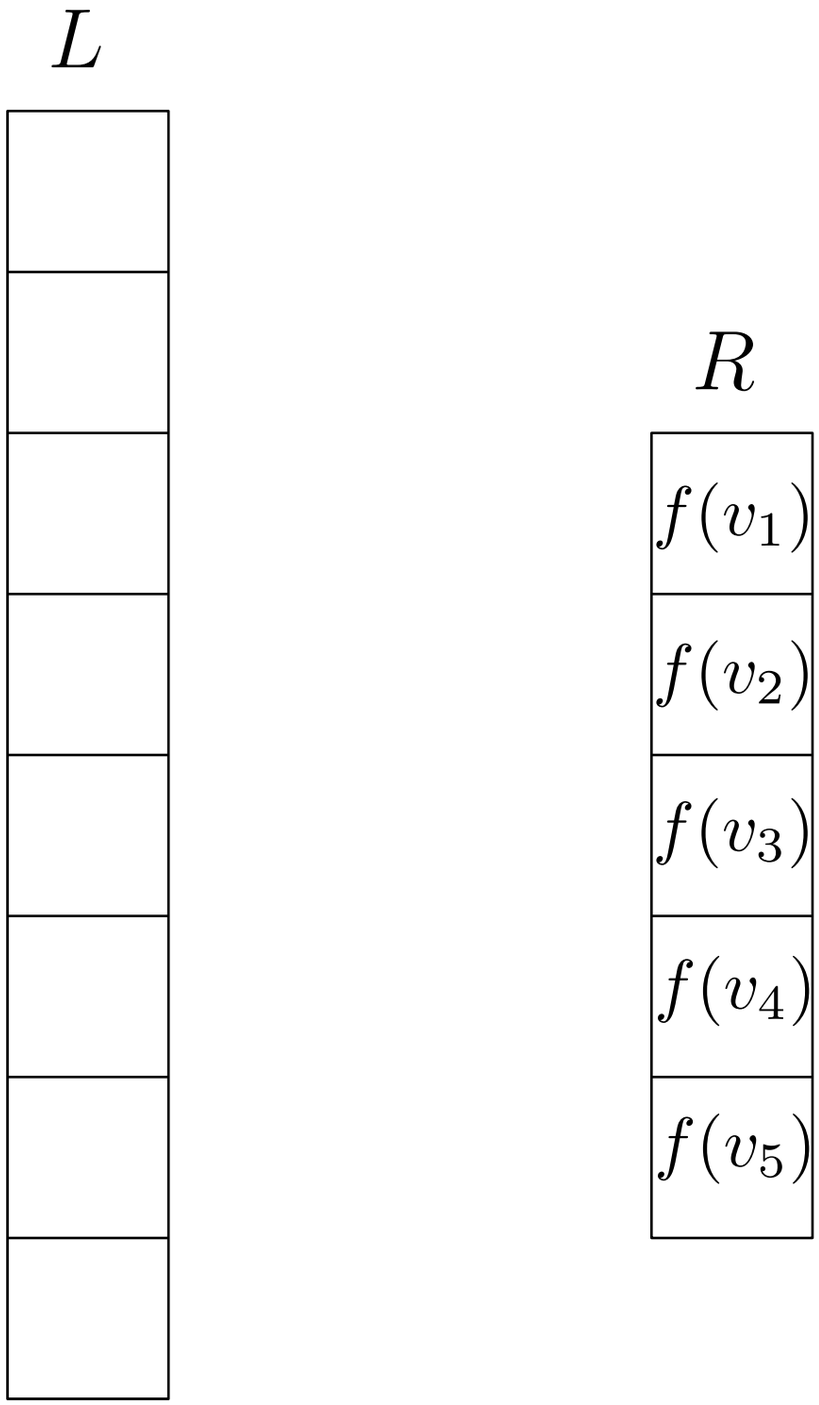

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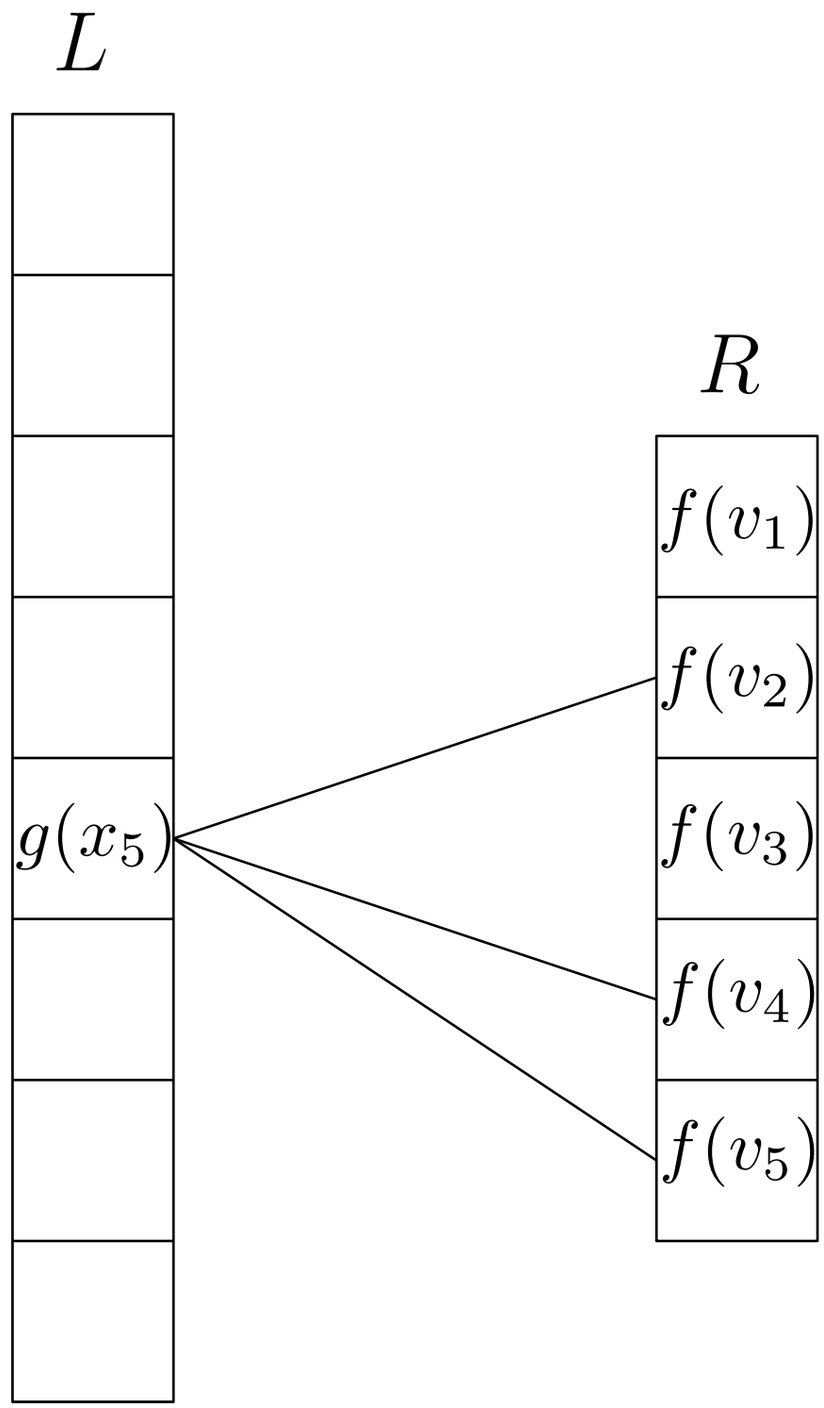

$$g(x_5) = f(v_2) + f(v_4) + f(v_5) \bmod p$$

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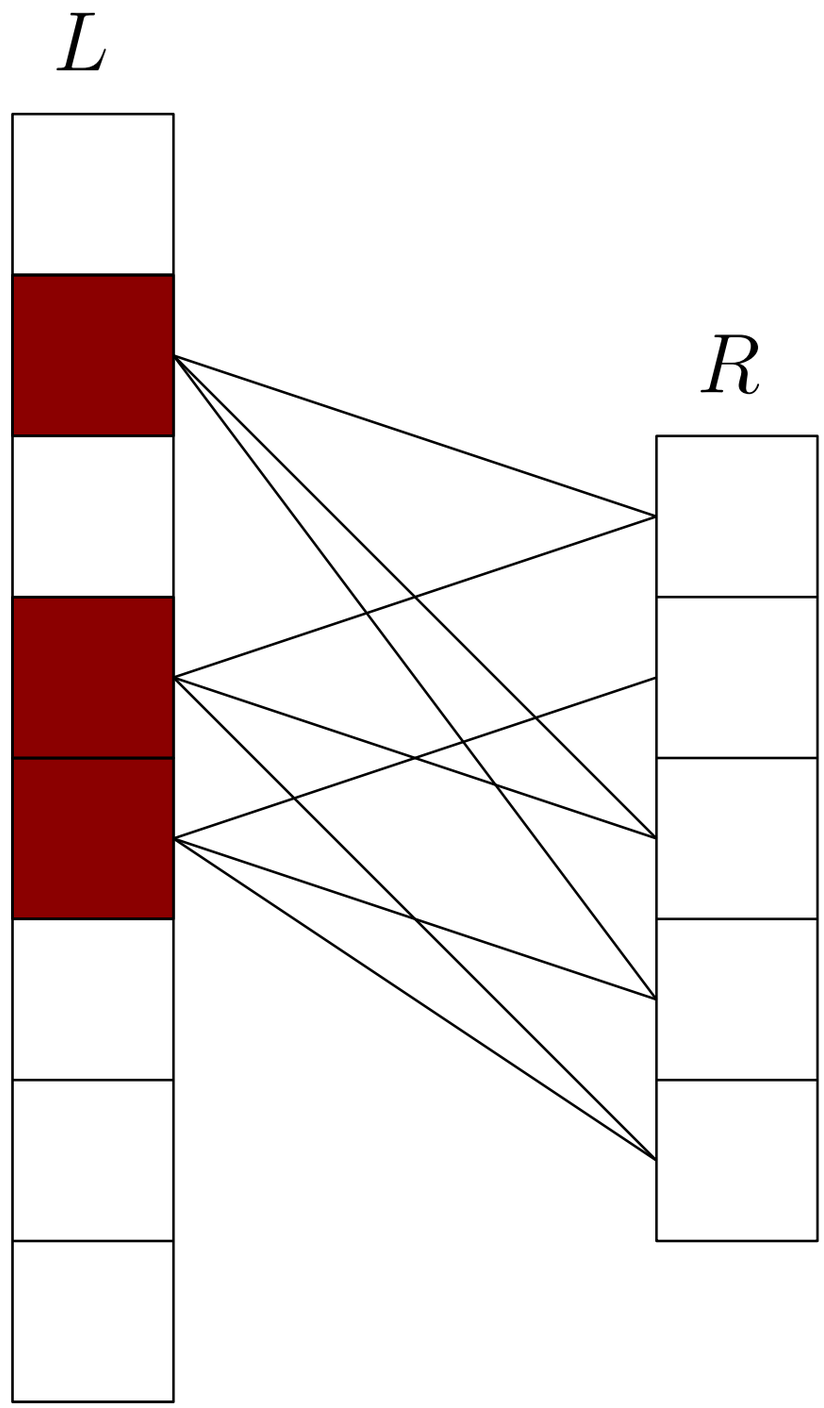

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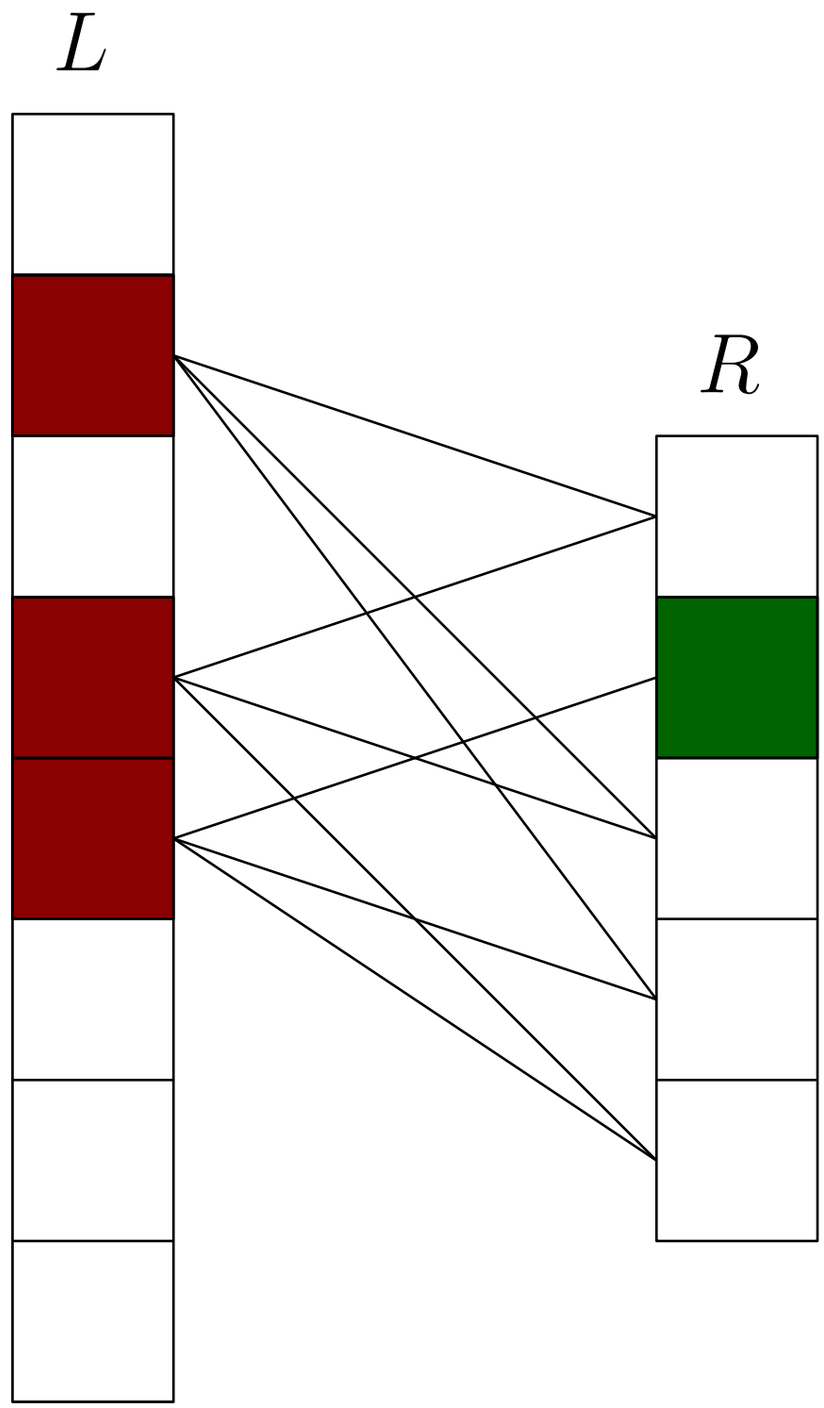

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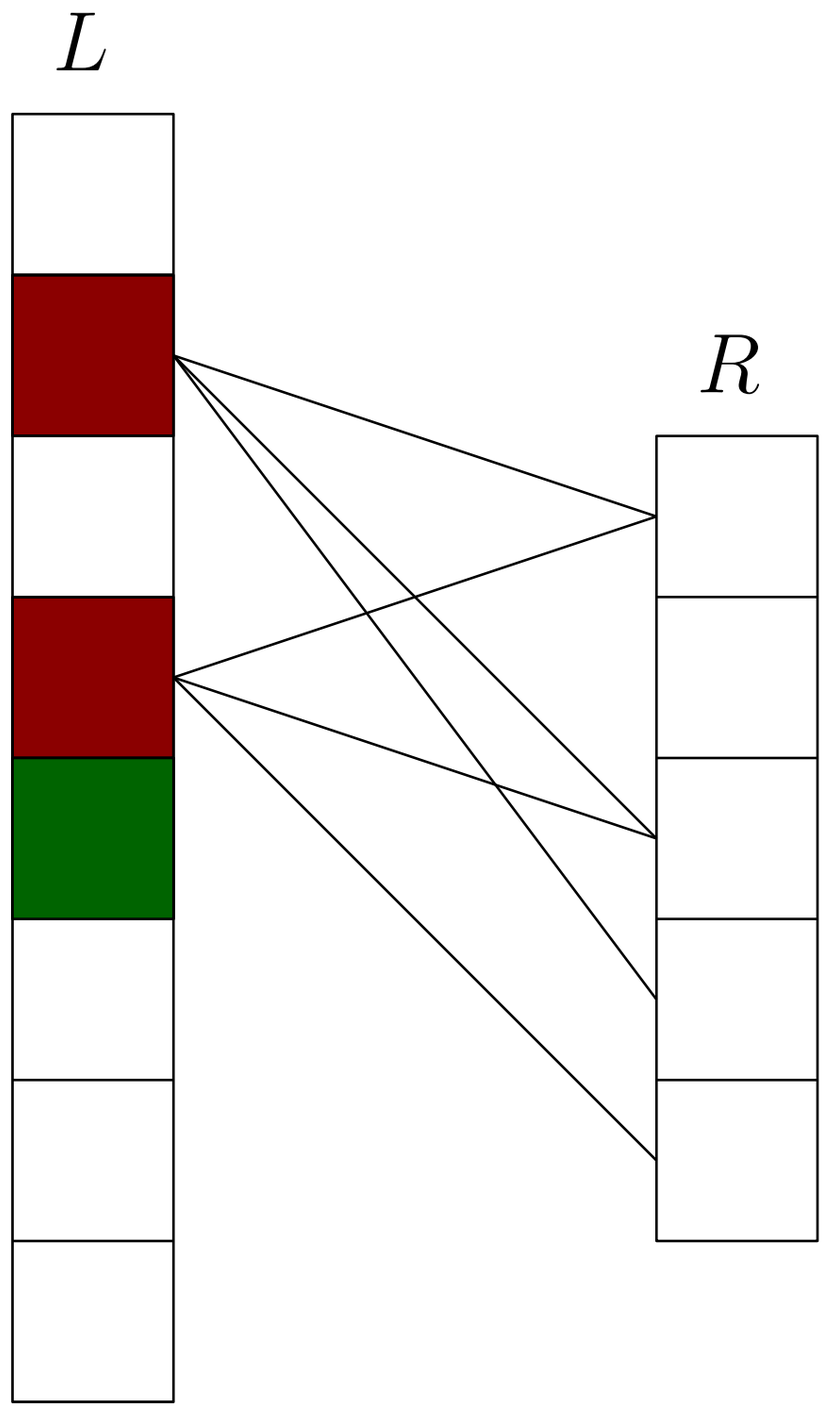

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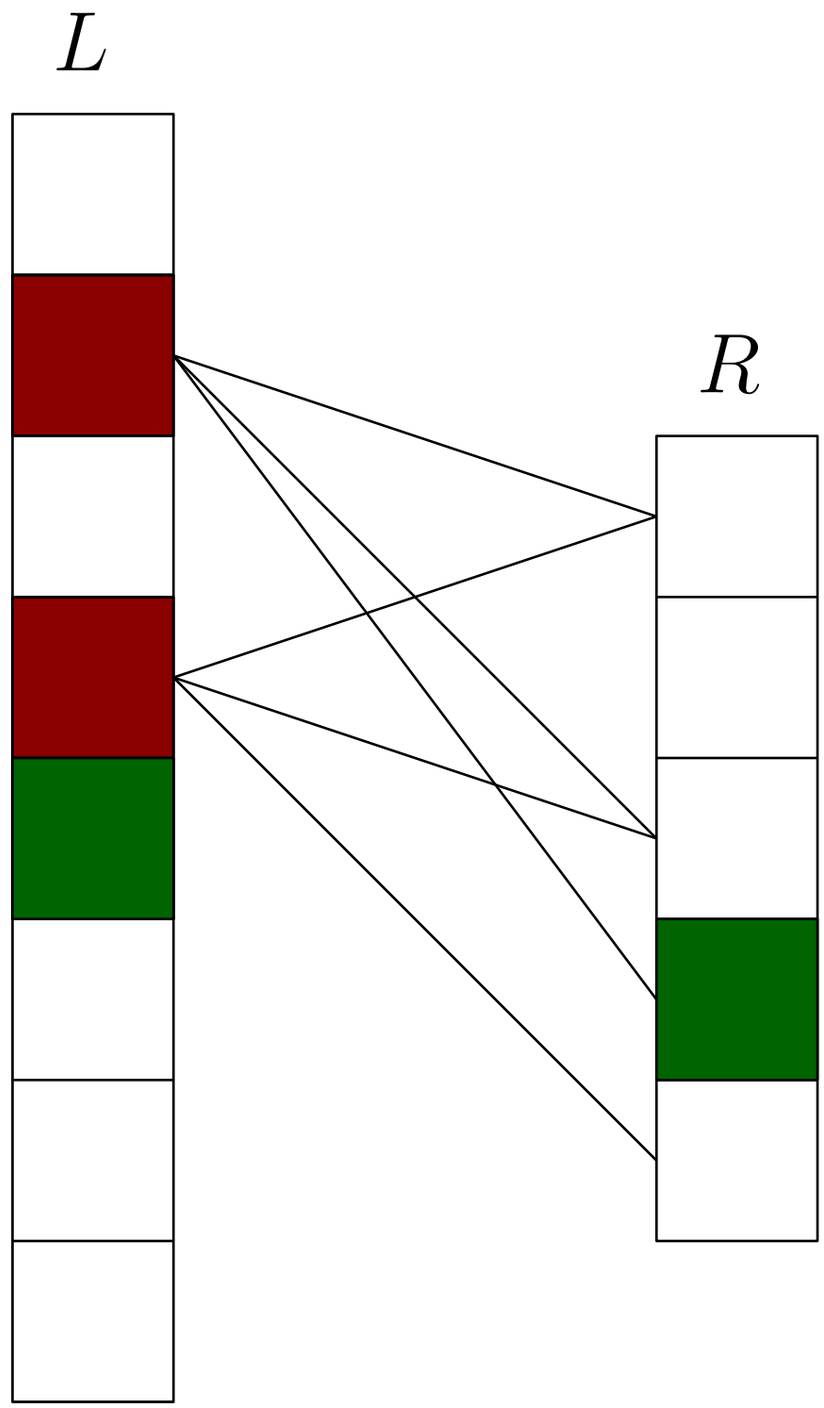

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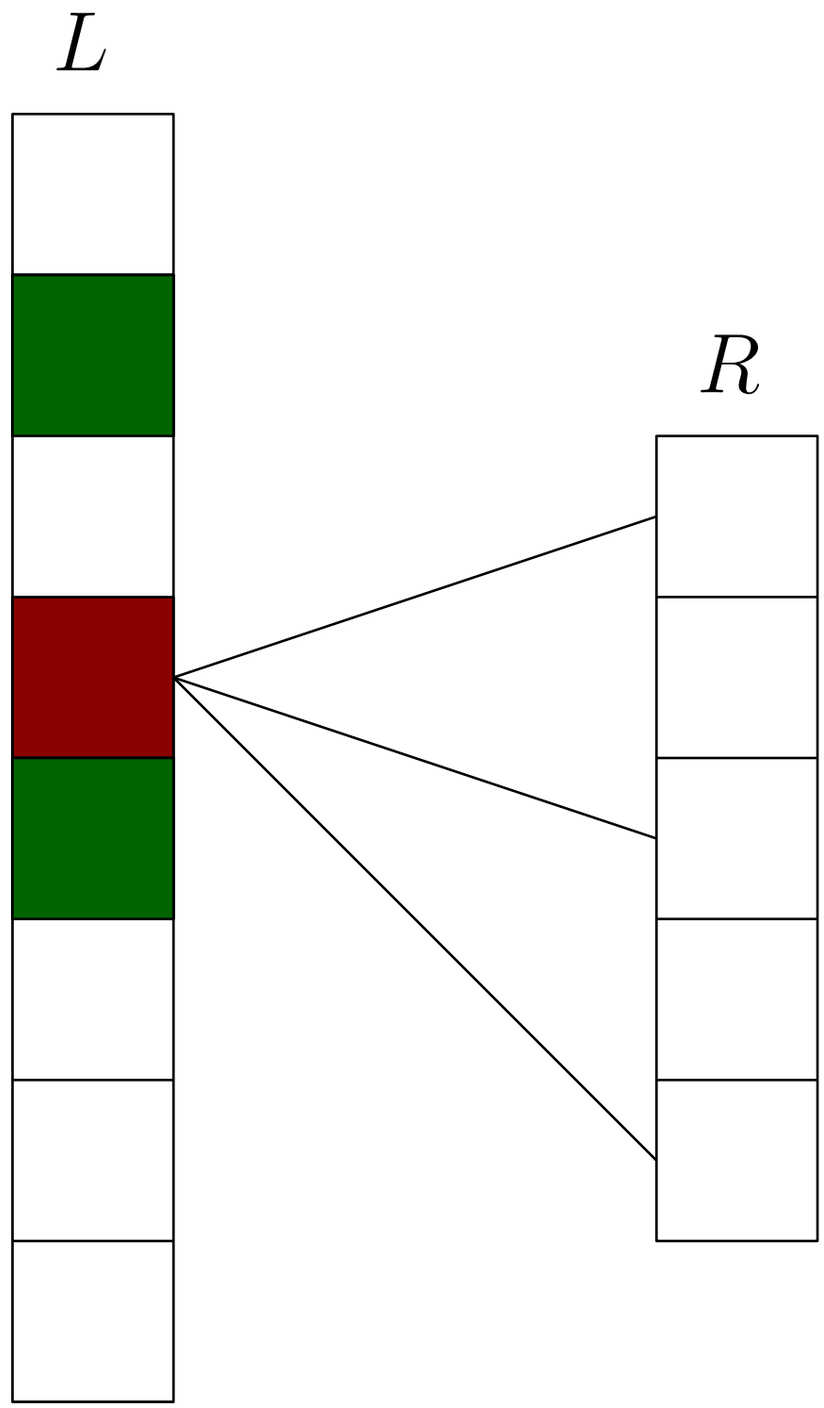

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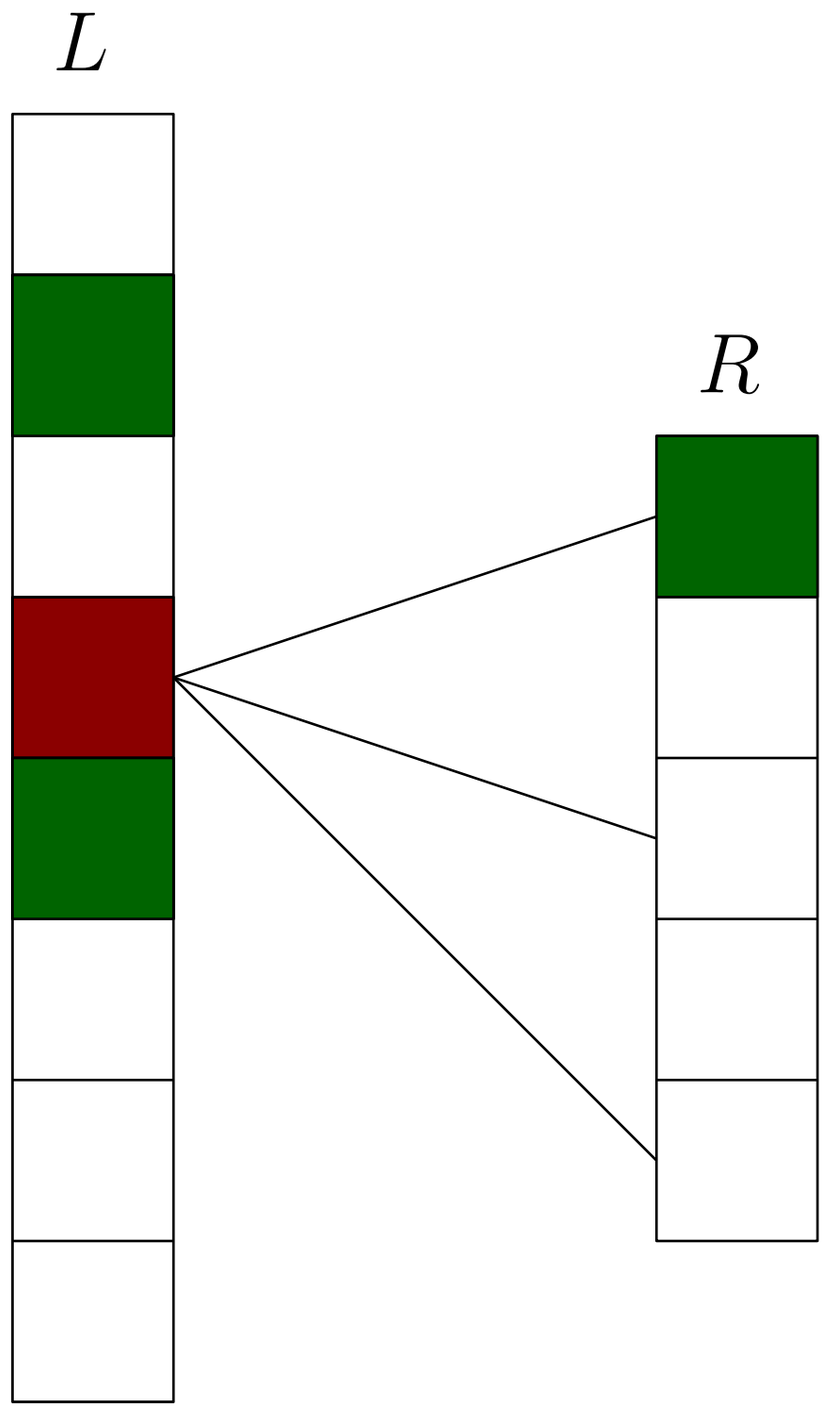

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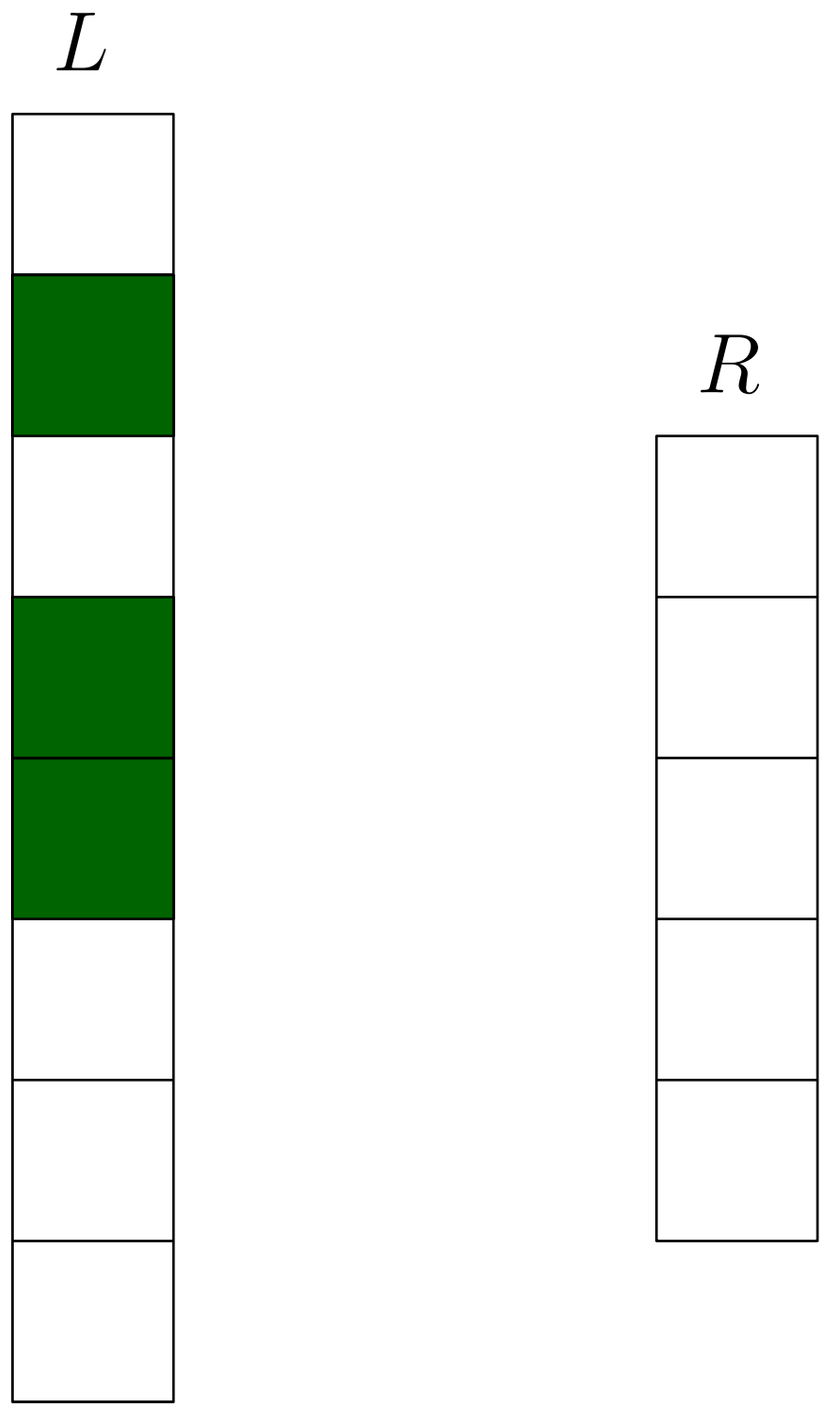

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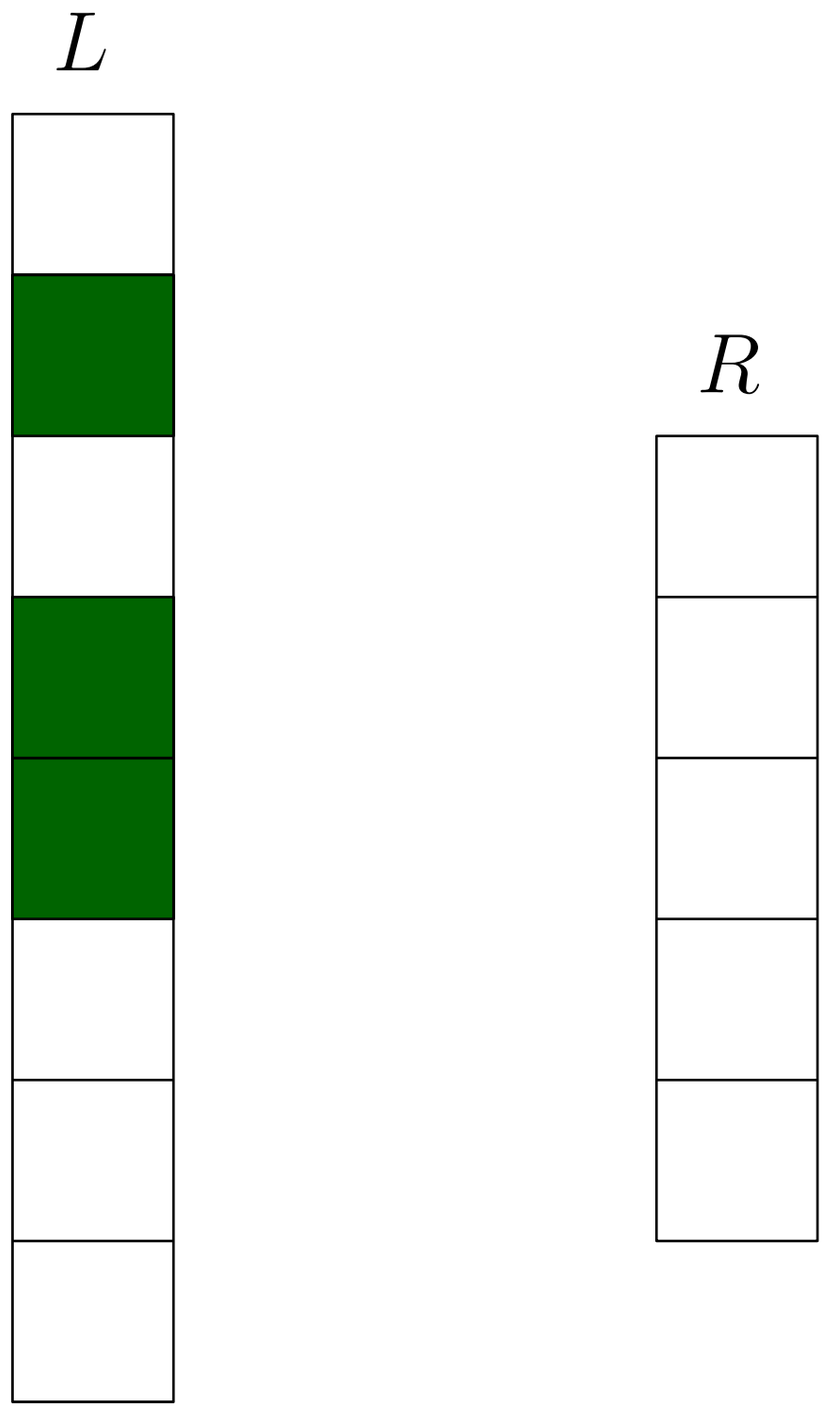

Claim:

- There exists k-unique Γ with
  - $|R| = O(k \log k)$
  - $|L| = (\text{poly} \log k) \cdot |R|$
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Example construction

Initialization and evaluation of local hash functions:

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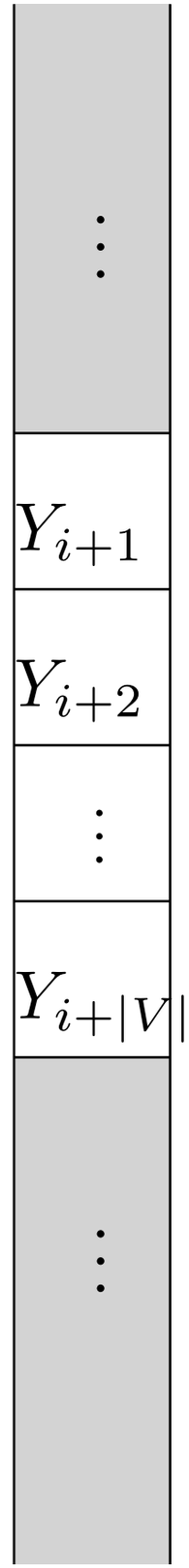

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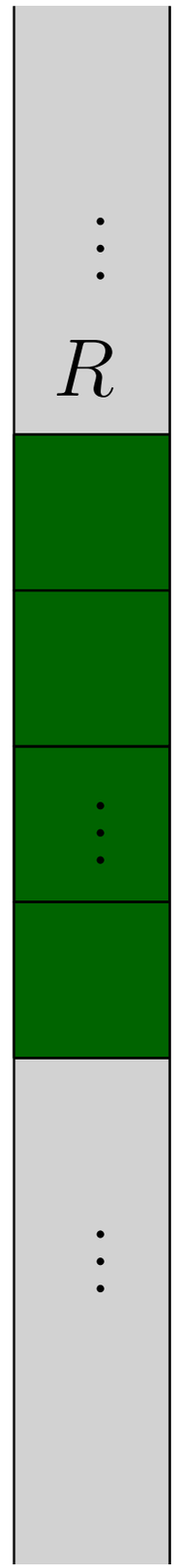

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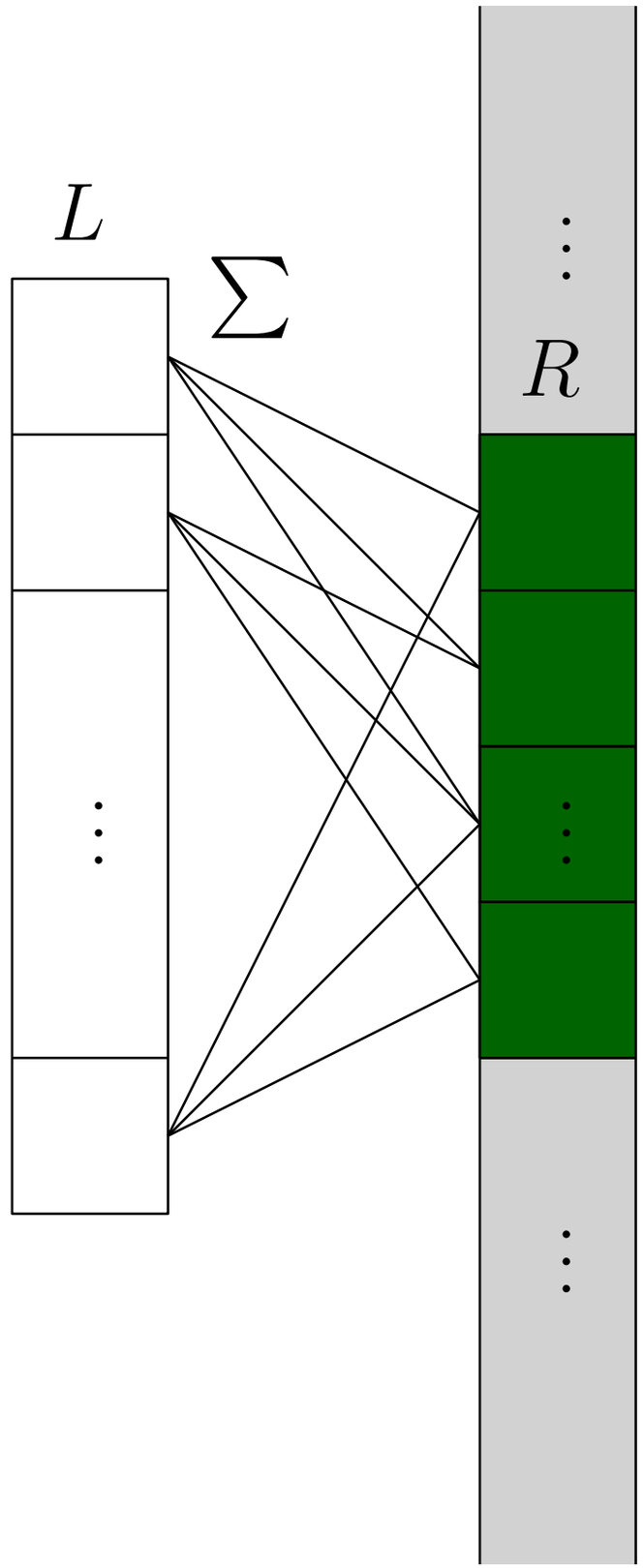

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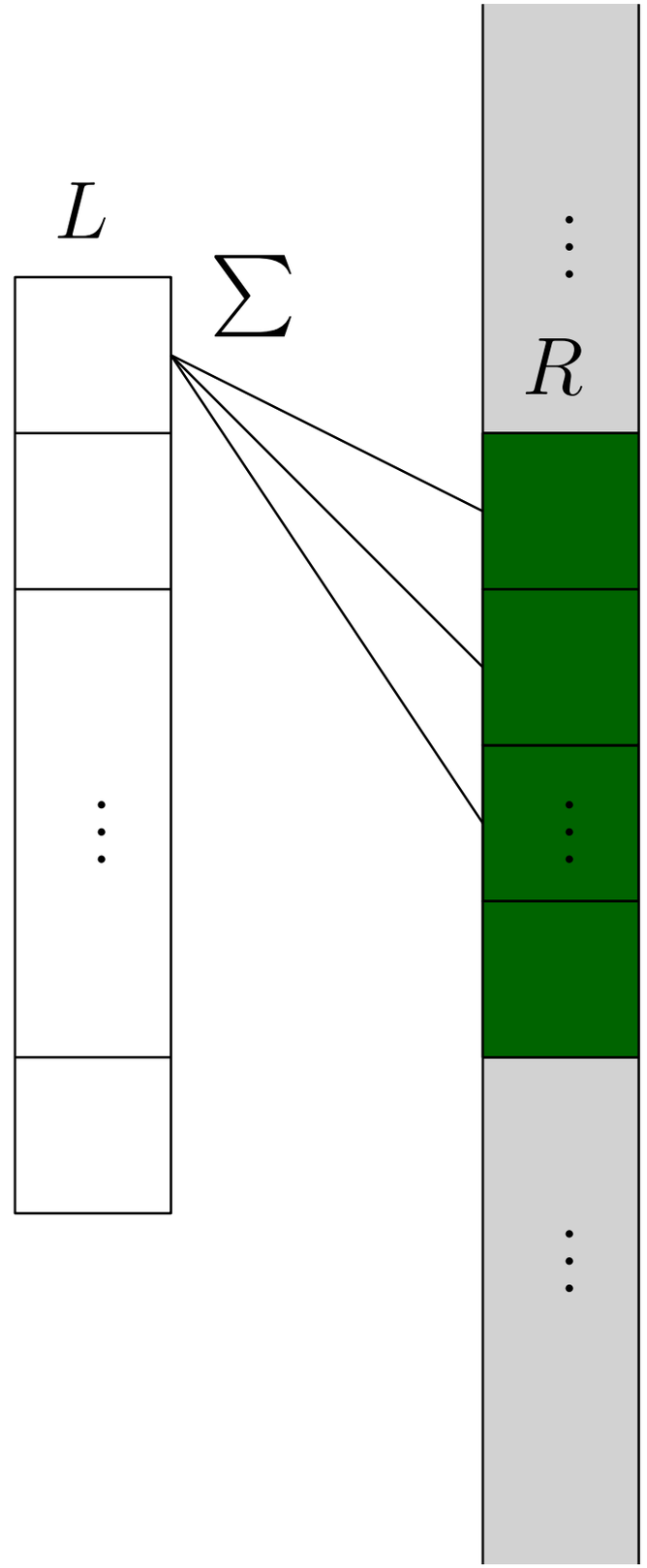

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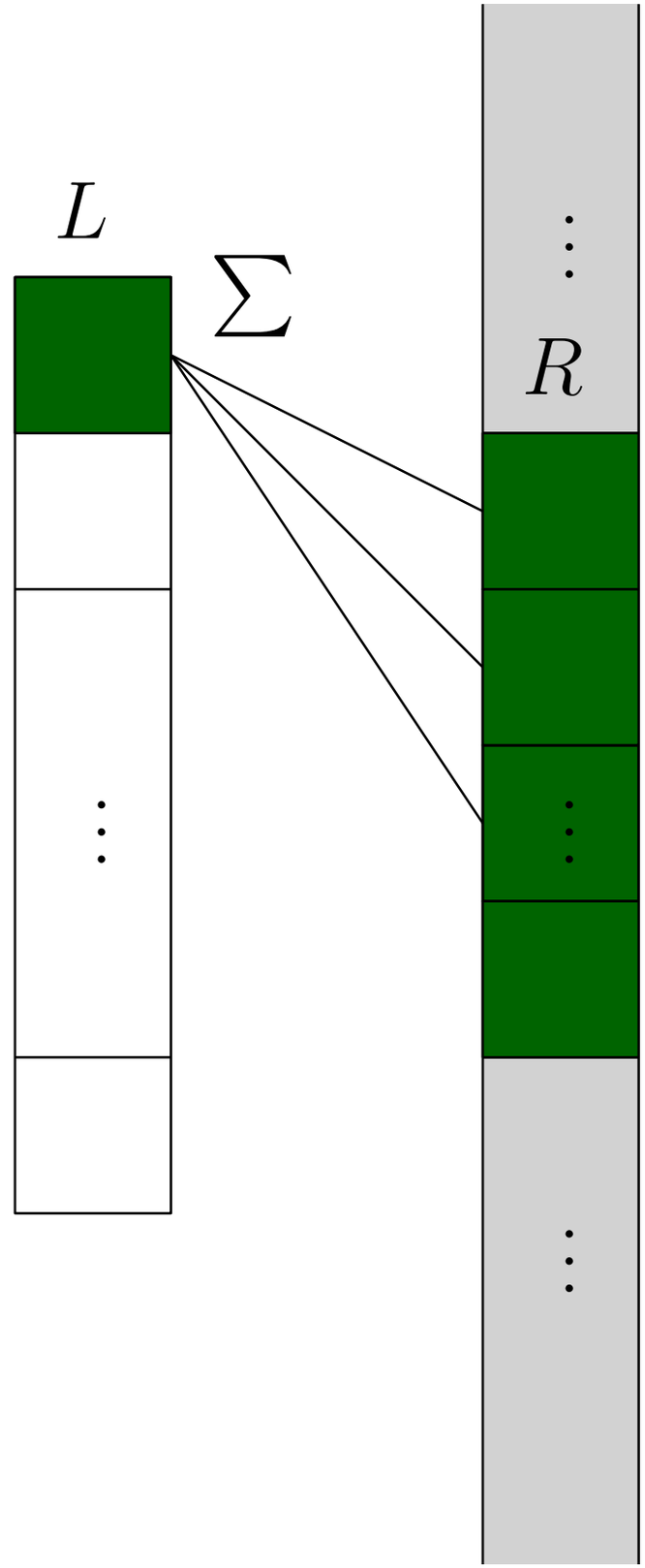

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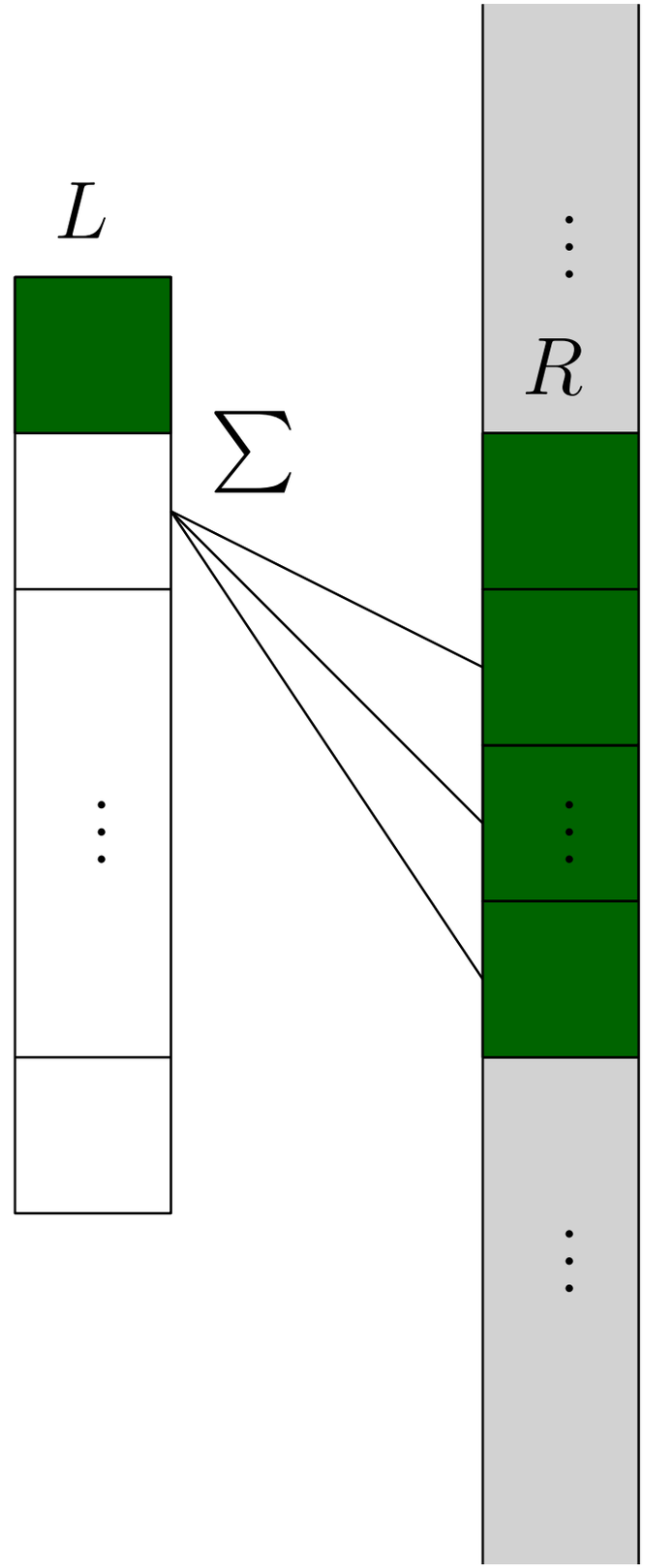

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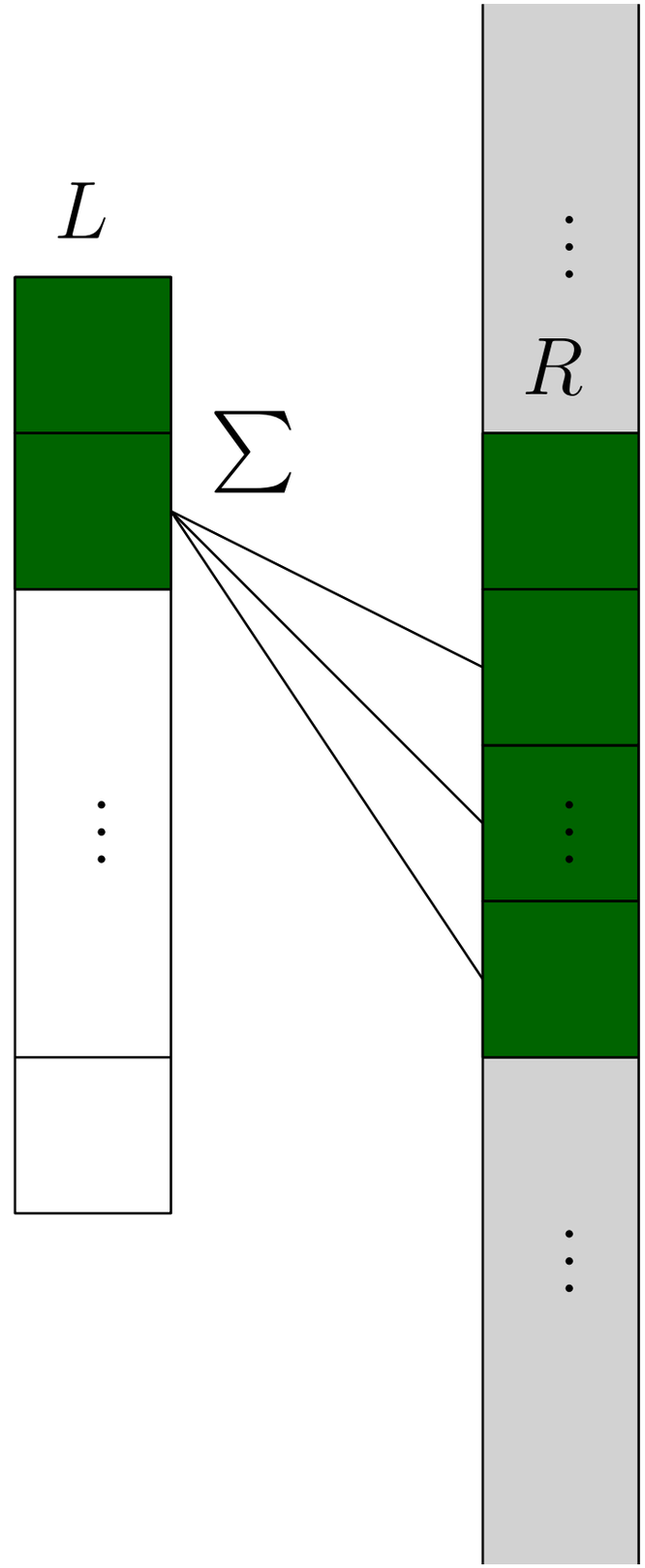

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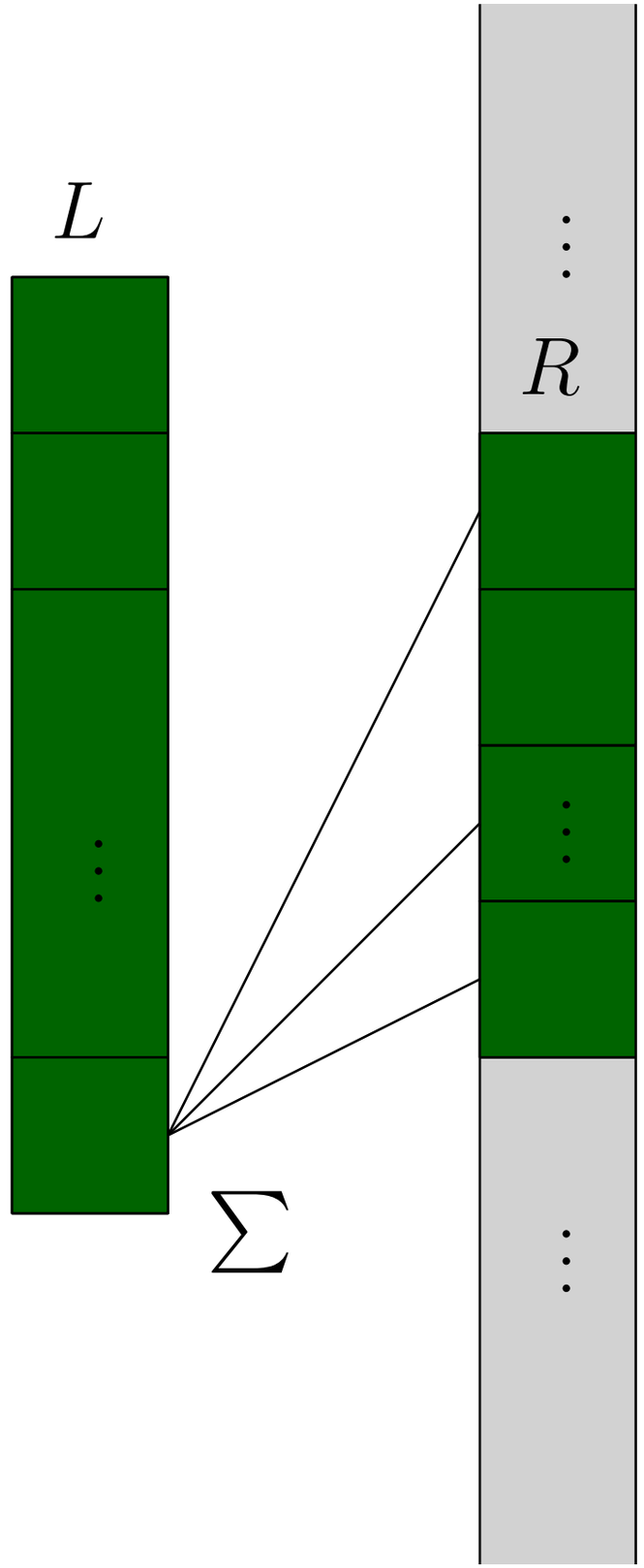

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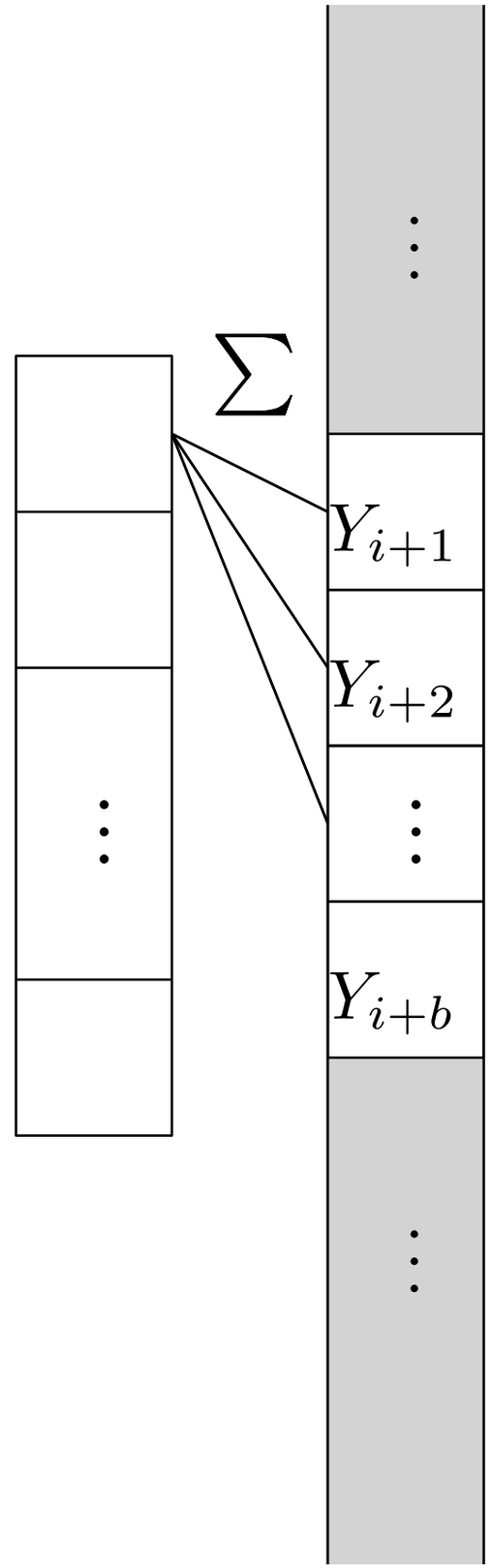

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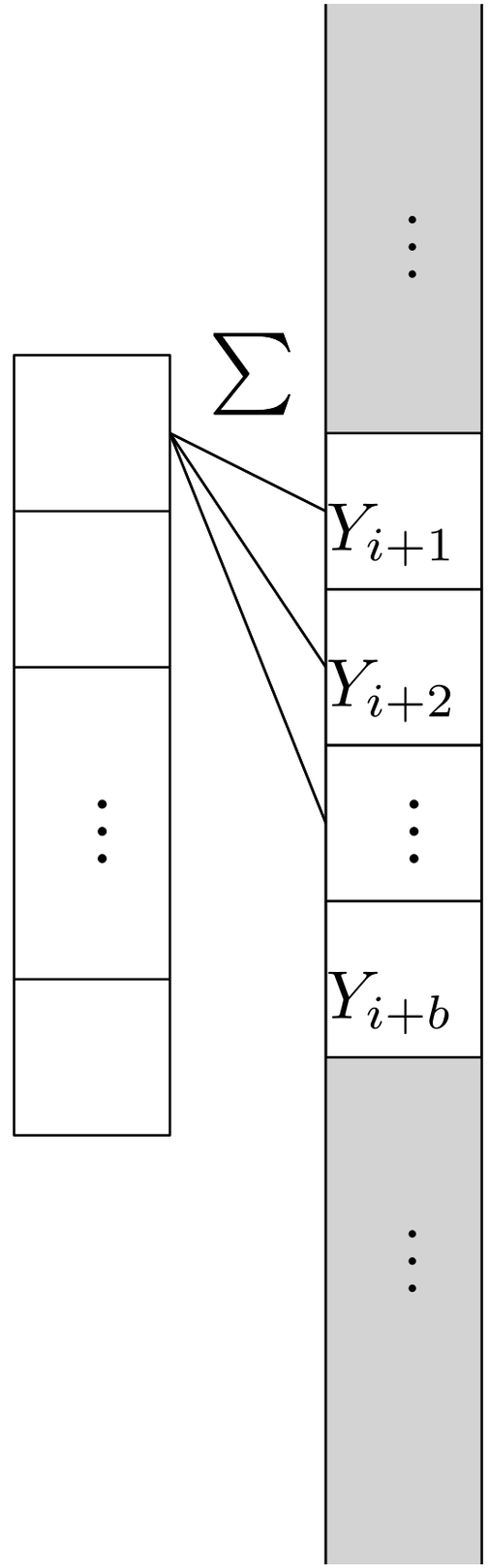

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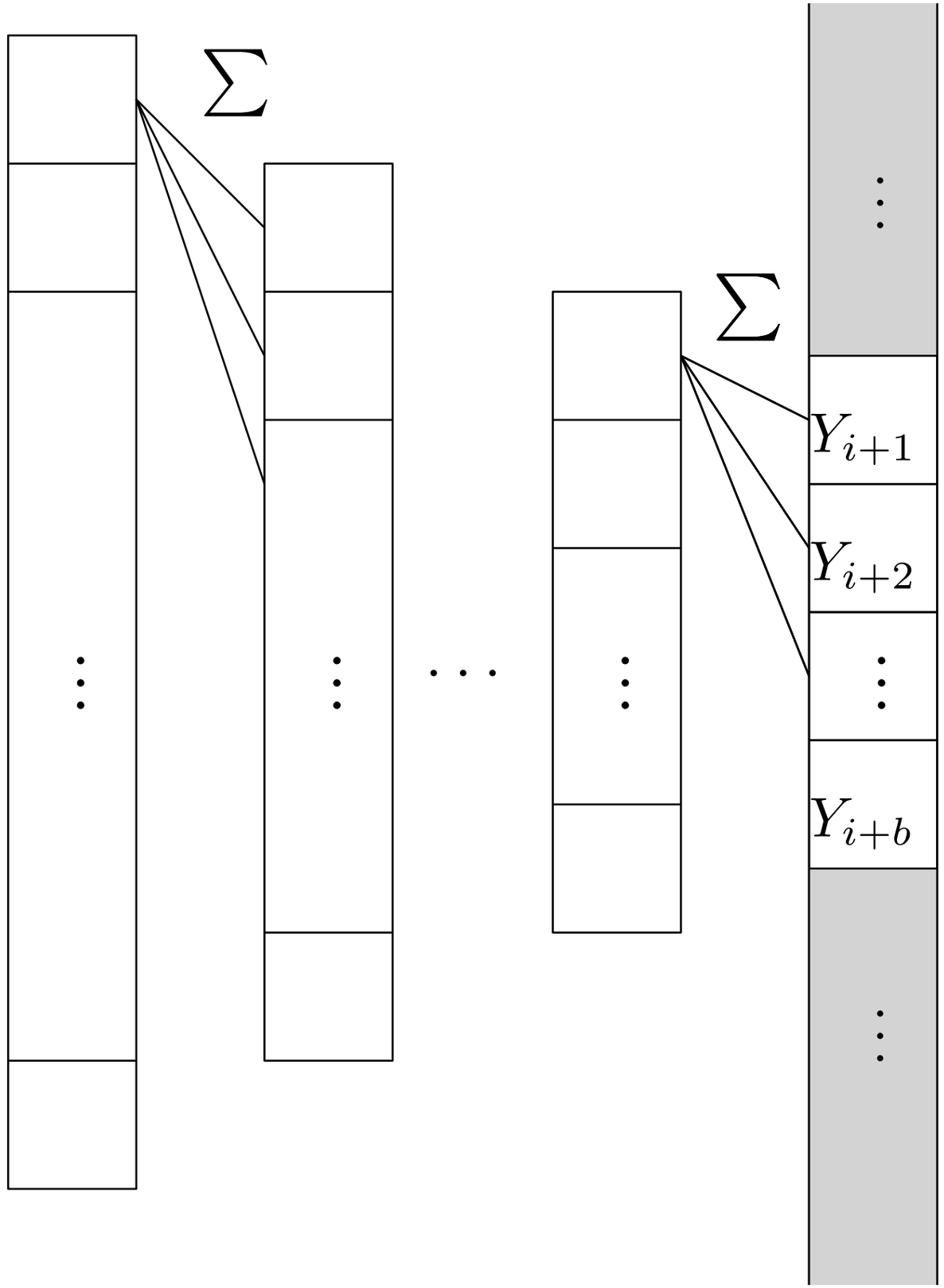

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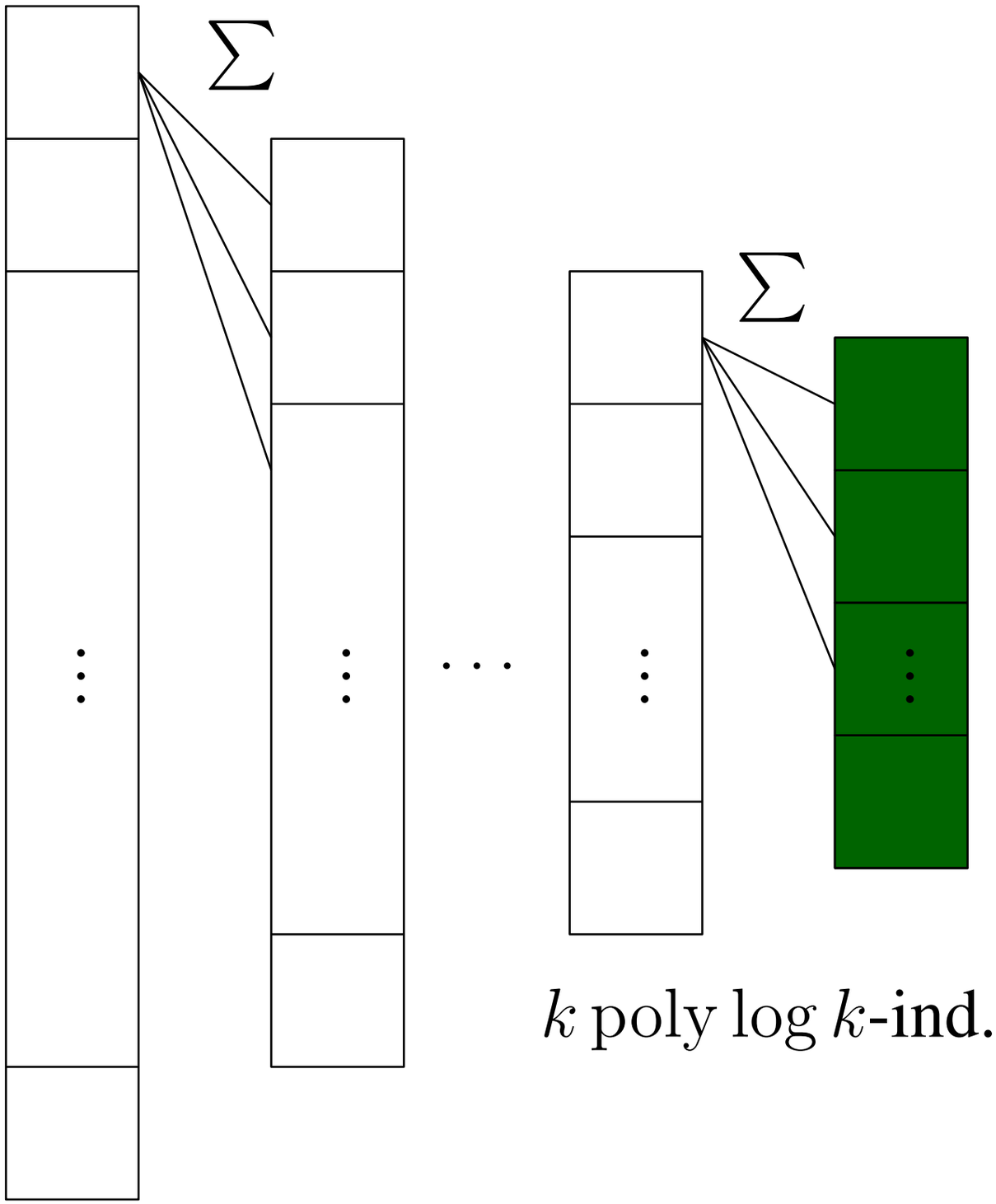

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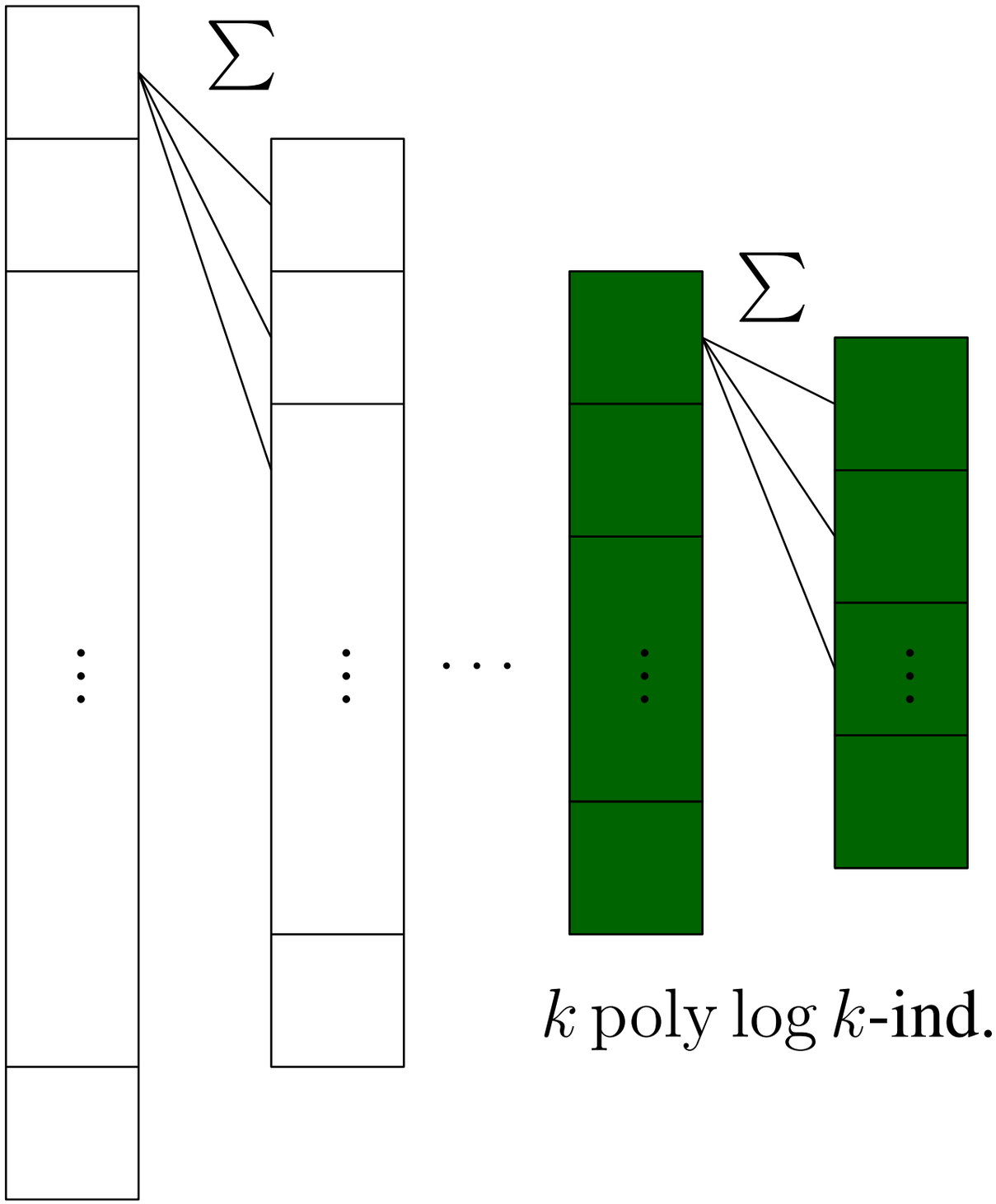

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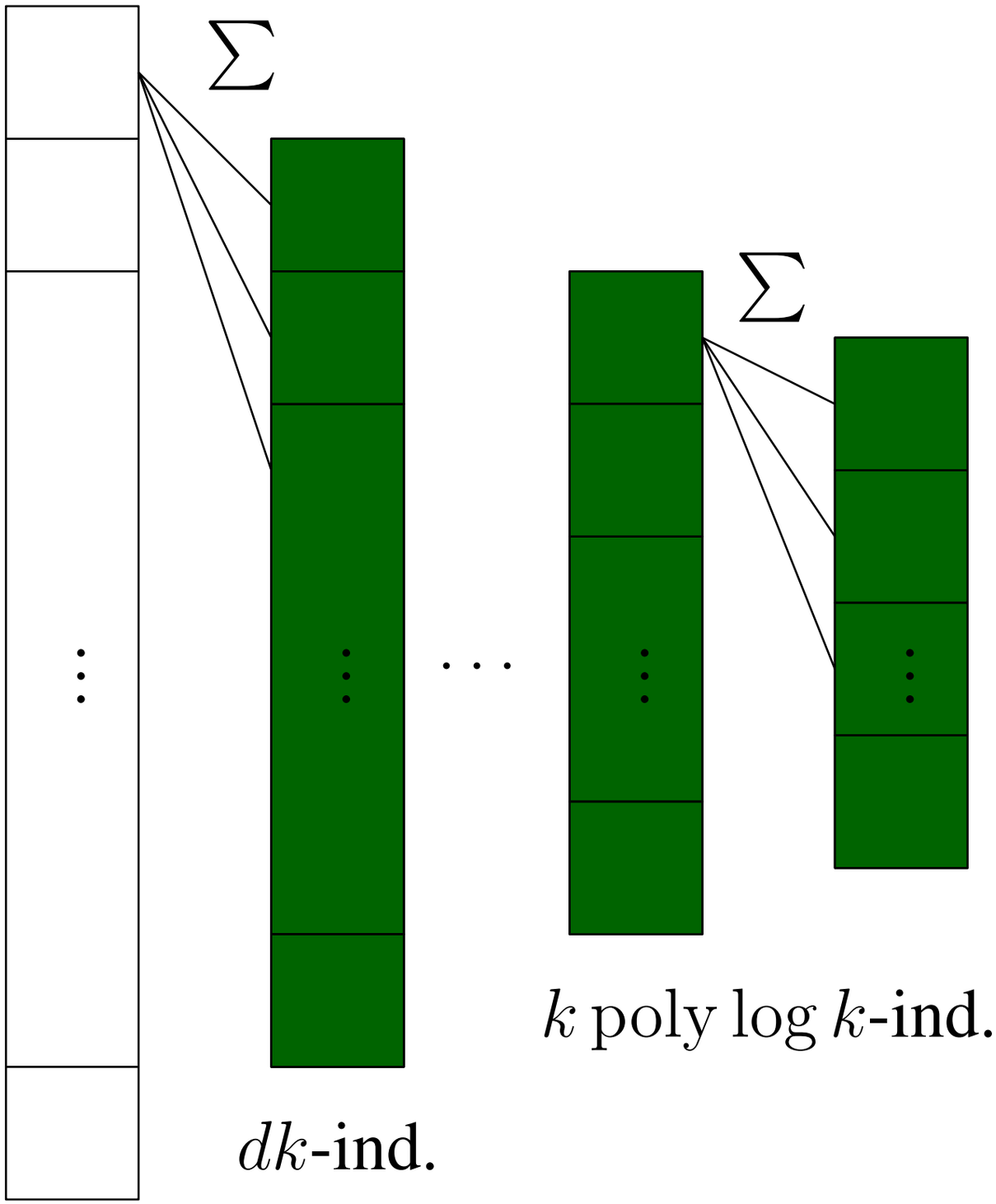

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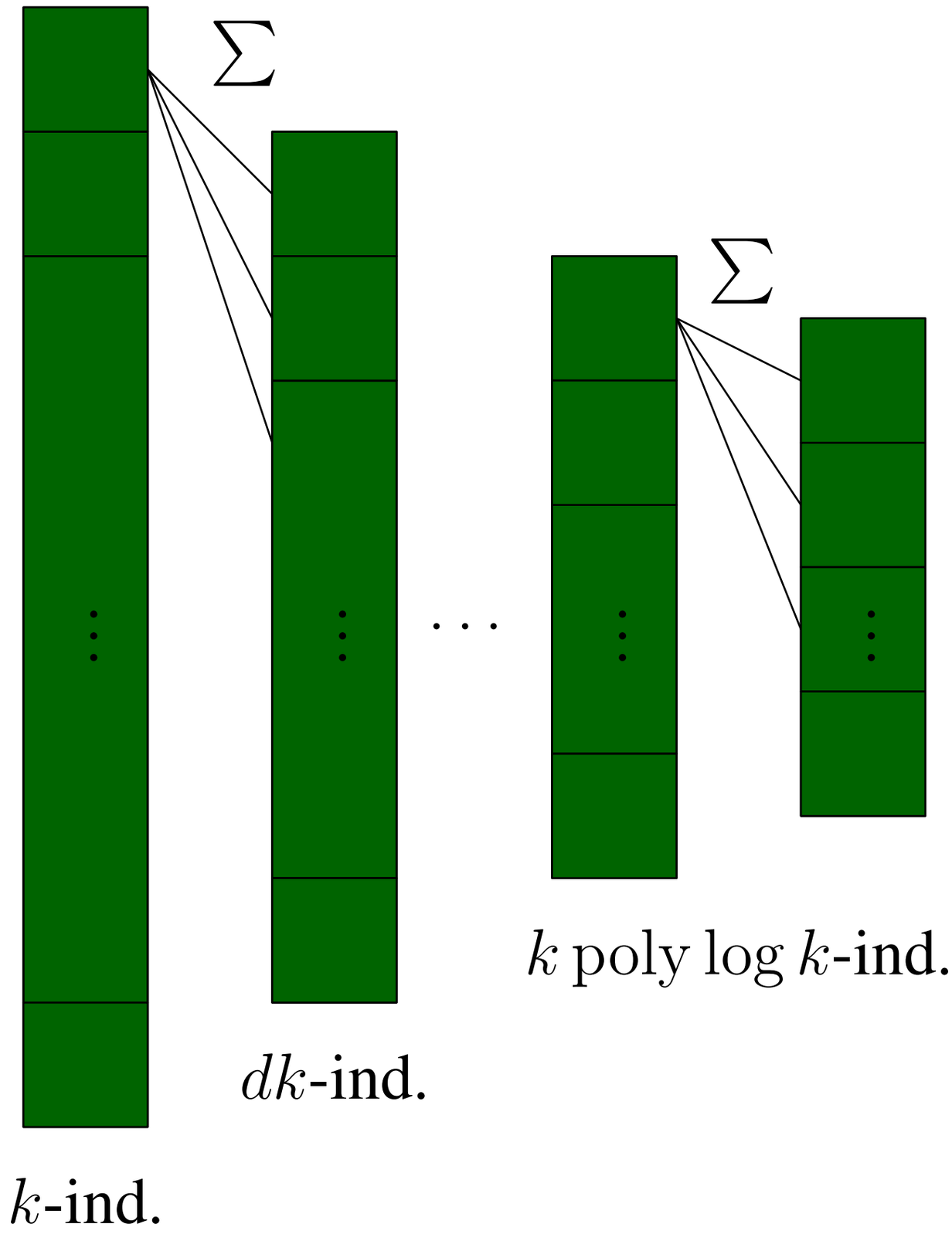

MILLIONS OF 64-BIT VALUES GENERATED PER SECOND, RANDOMIZED CONSTRUCTION

k	**Polynomial hash fct.**	**Multipoint [GM'10]**	**Our generator**
2^5	5.650	4.115	66.667
2^{10}	0.169	2.227	40.000
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- Constant time generation with $O(k)$ space

Multipoint evaluation

- A k-independent polynomial hash function $h(x)$
- We can evaluate $h(x)$ in k arbitrary points $x_1, x_2, \ldots, x_k$ using $O(k \log^3 k)$ operations

$$h(x) = q(x)p(x) + r(x)$$

$$p_1(x) = \prod_{i=1}^{k/2}(x - x_i) \qquad p_2(x) = \prod_{i=(k/2)+1}^{k}(x - x_i)$$

$$r_1(x) = h(x) \bmod p_1(x) \qquad r_2(x) = h(x) \bmod p_2(x)$$

$$r_1(x_i) = h(x_i) \text{ for } 1 \leq i \leq k/2$$

- Faster algorithms exist if:
  - $x_1, x_2, \ldots, x_k$ lie in arithmetic/geometric progression
  - $\mathbb{F}$ contains a highly composite kth root of unity

New results

- We only require sequential evaluation of Γ
- Tabulating Γ requires at least $|R| = \Omega(k \log^{2+\varepsilon} k)$ words
- Constant time generator for k-unique $\Gamma : L \to R^d$ using space $O(k \log^{\varepsilon} k)$
- New recursive constructions of k-independent hash functions that come close to Siegel's lower bound