Succinct Data Structures for Retrieval and Approximate Membership

Martin Dietzfelbinger
Technische Universität Ilmenau

Joint work with Rasmus Pagh
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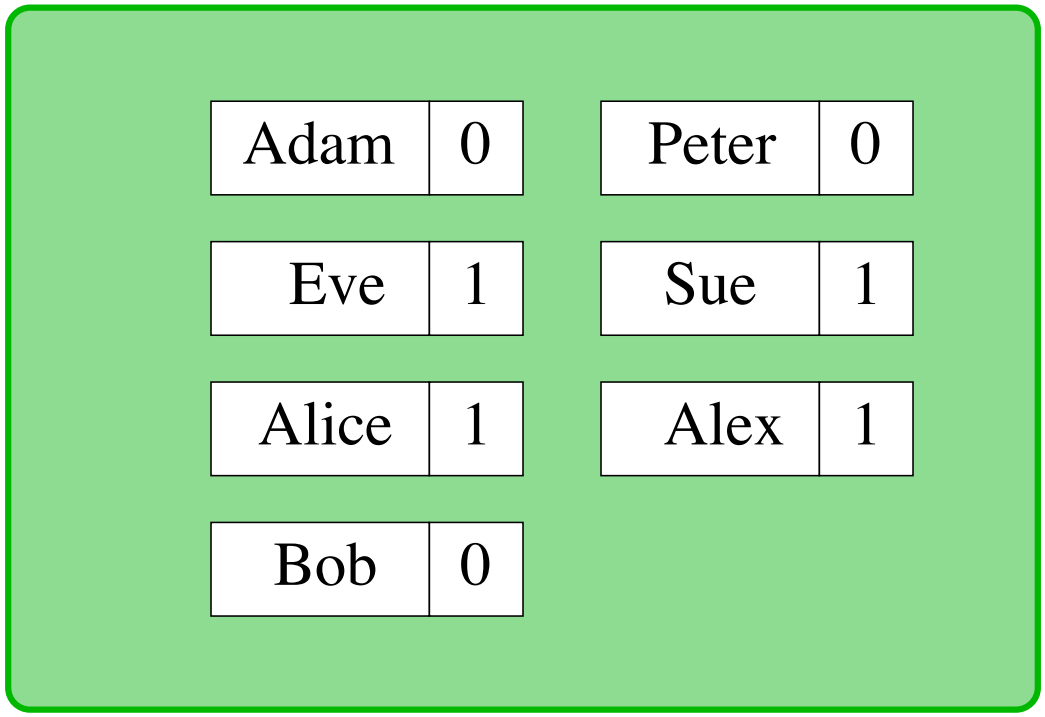
Retrieval



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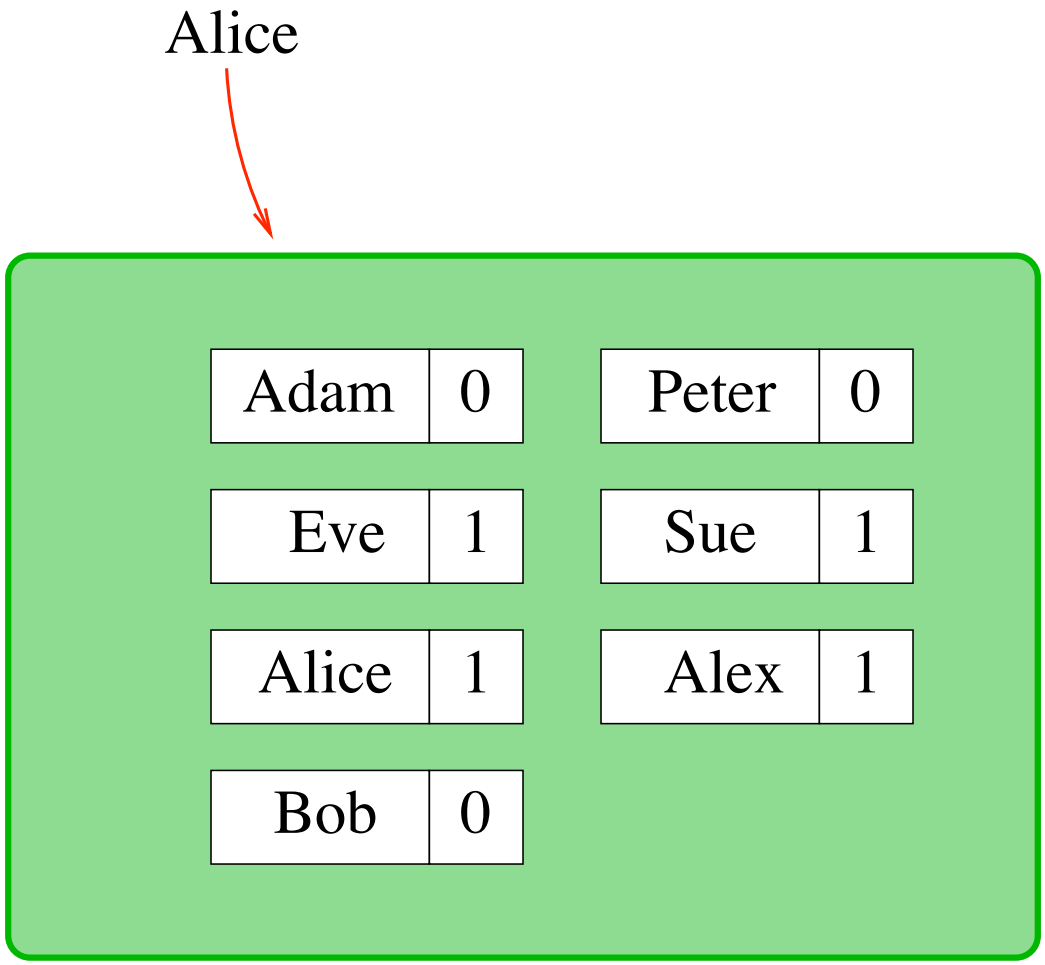

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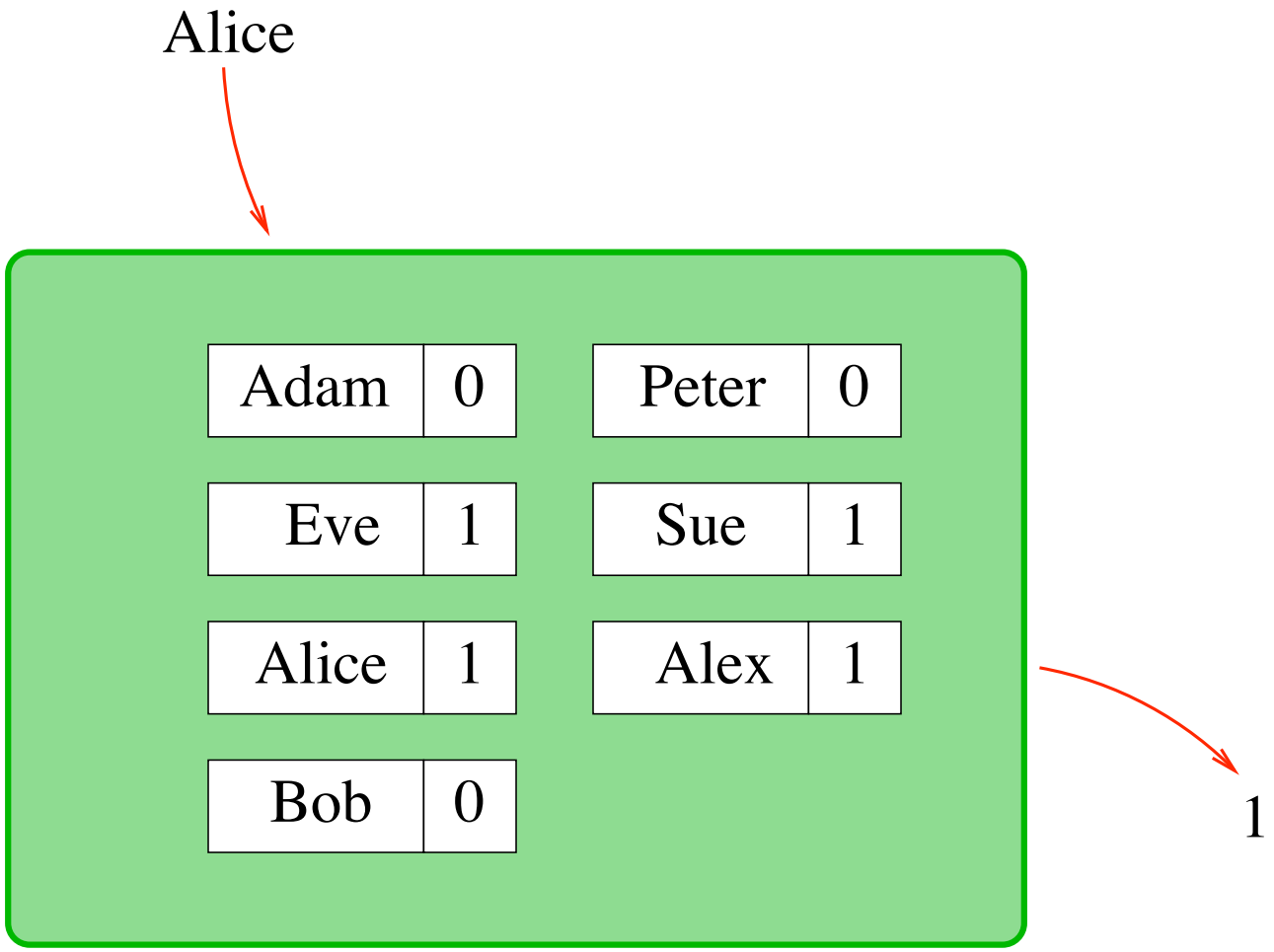

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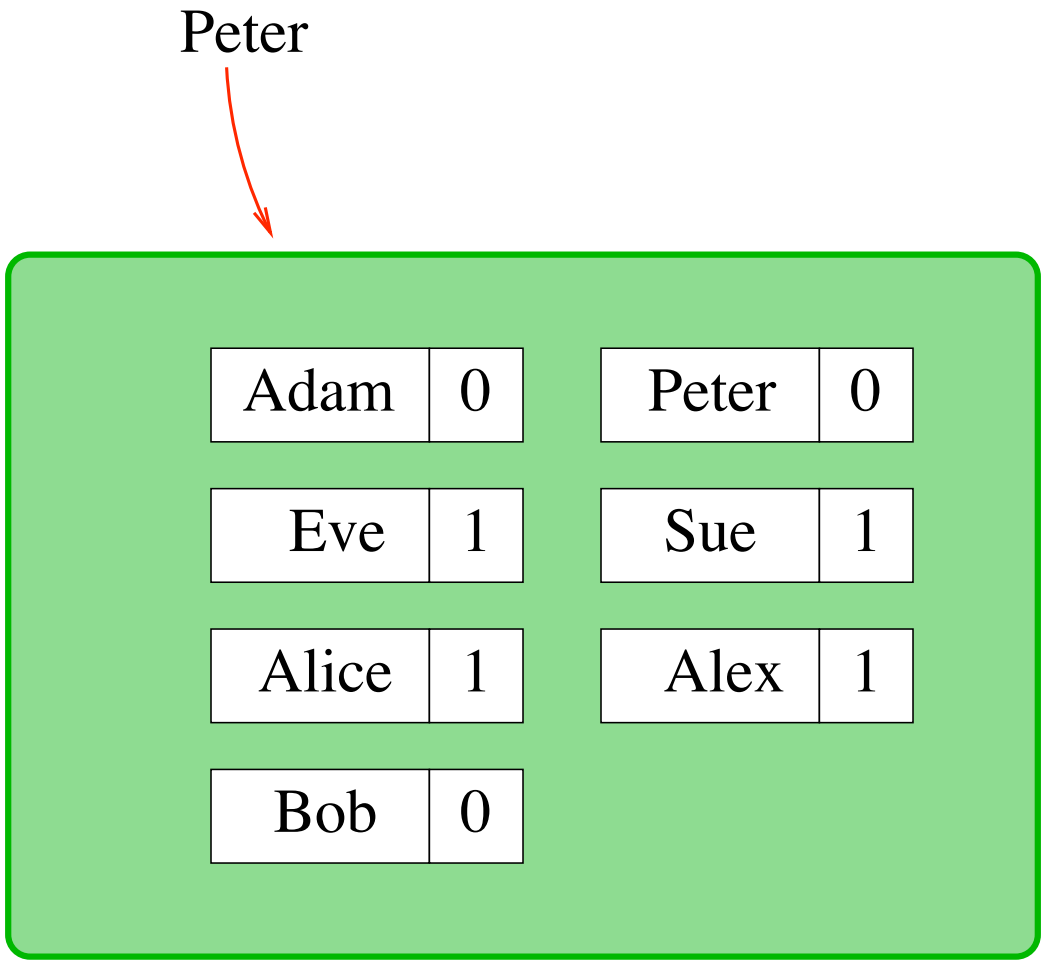

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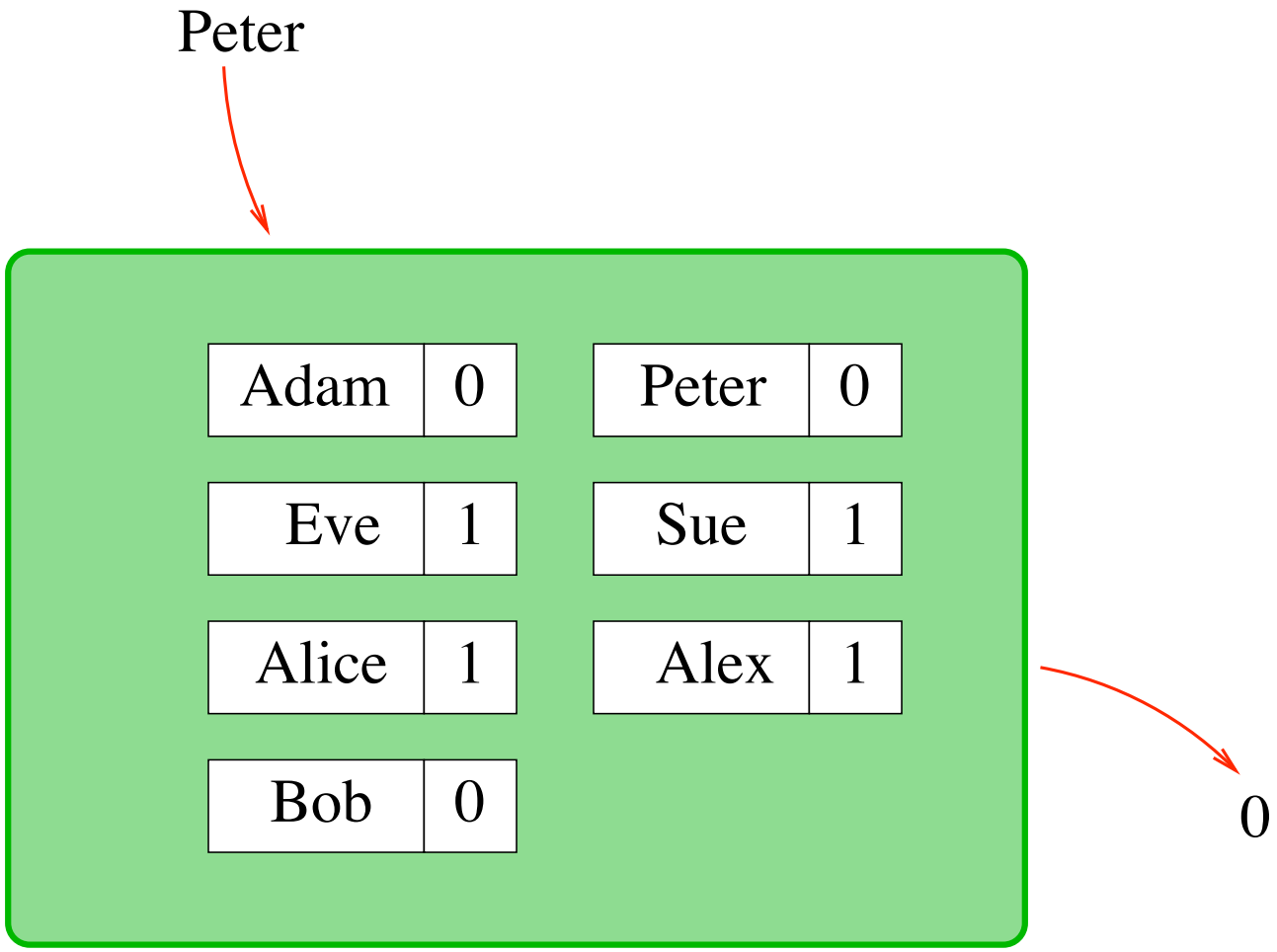

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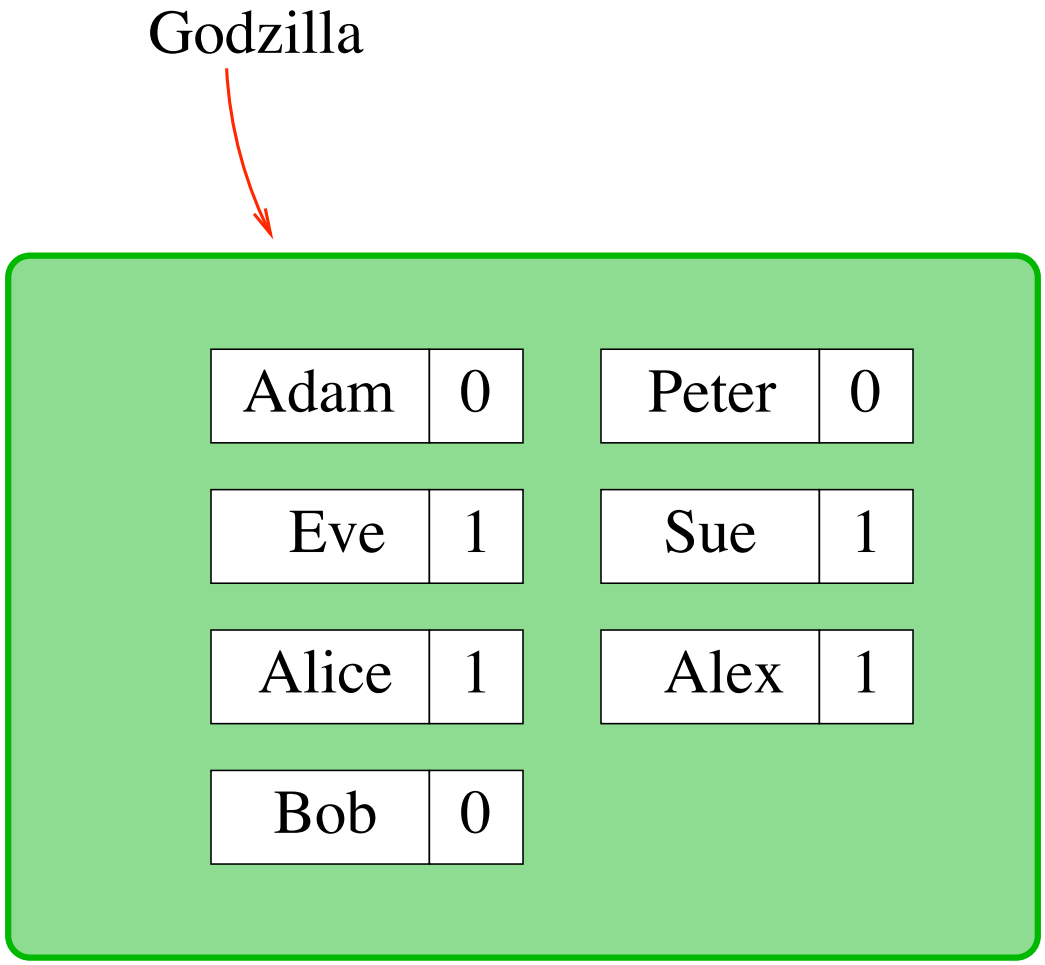

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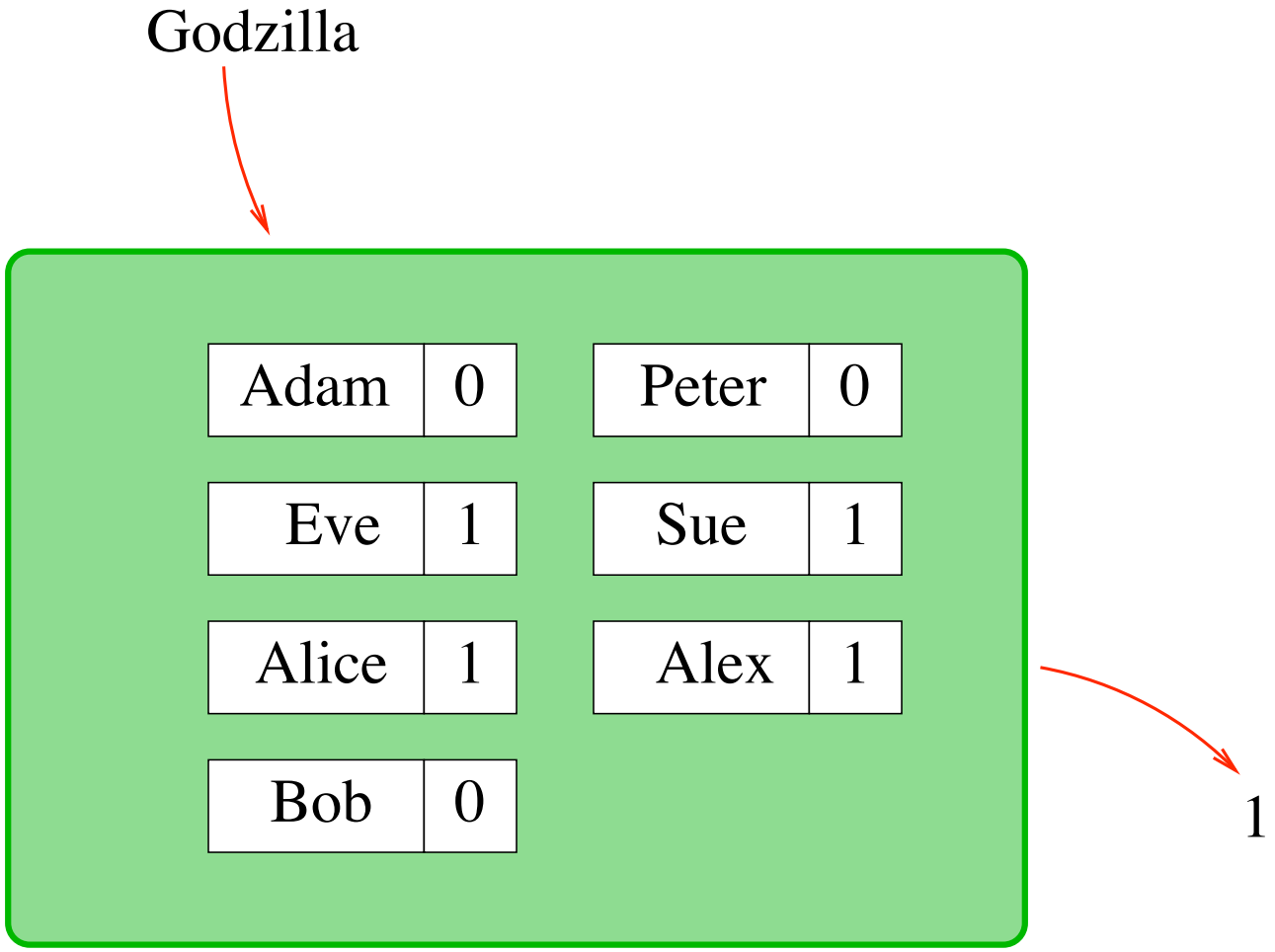

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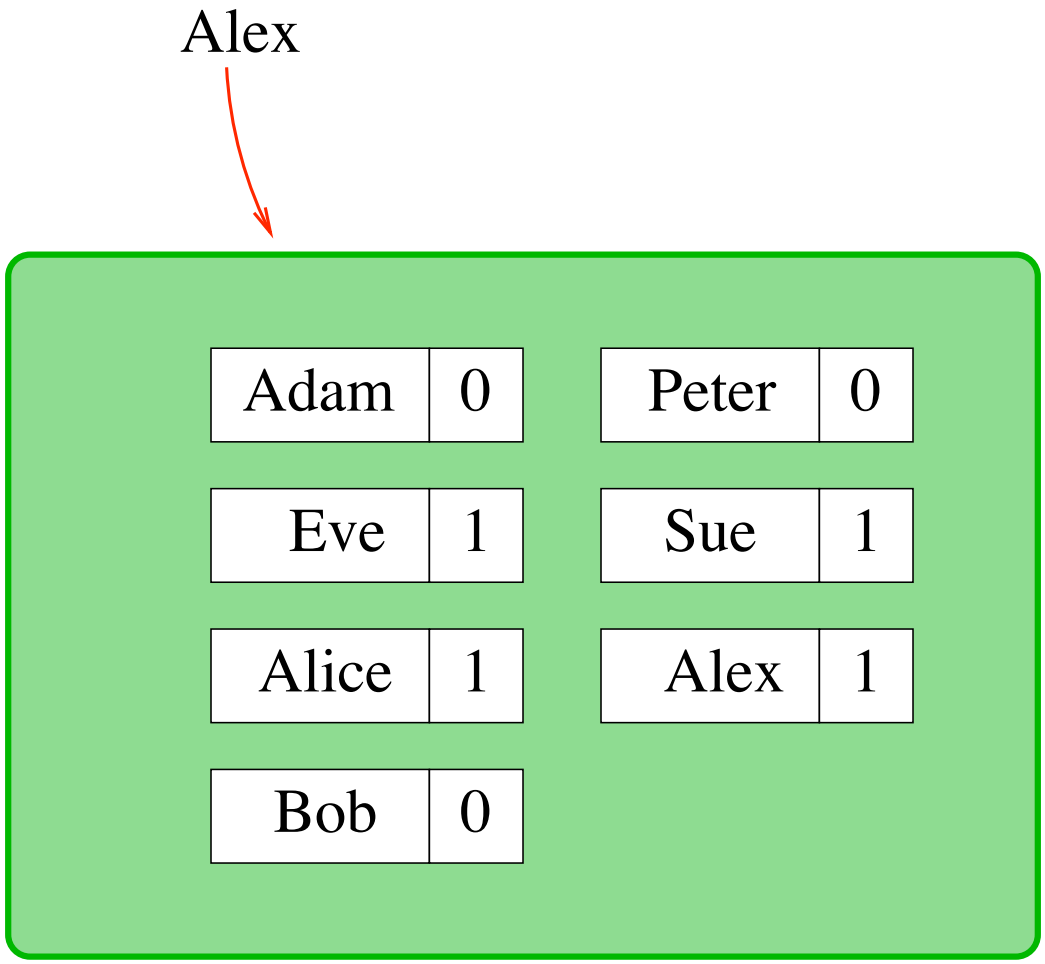

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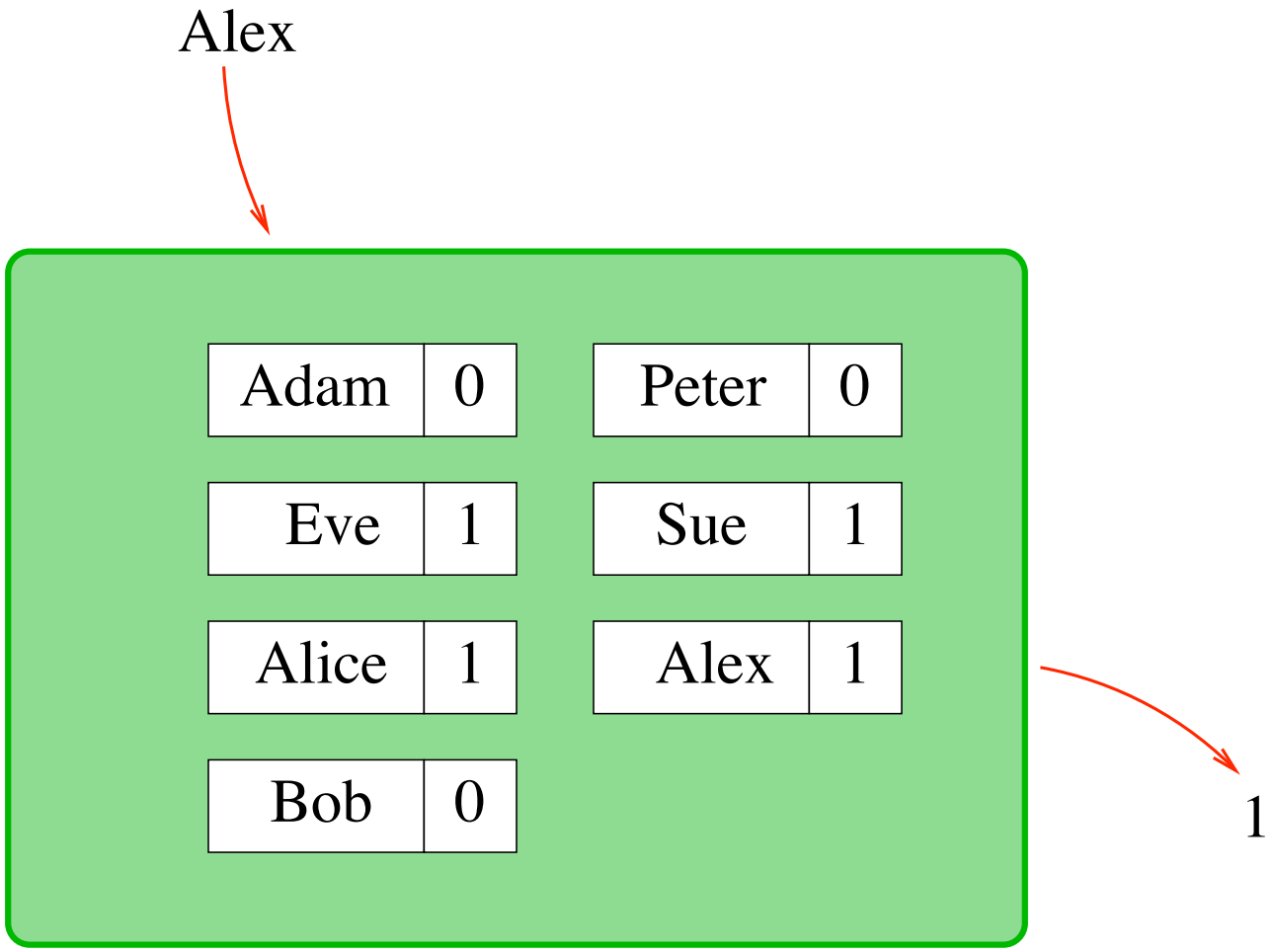
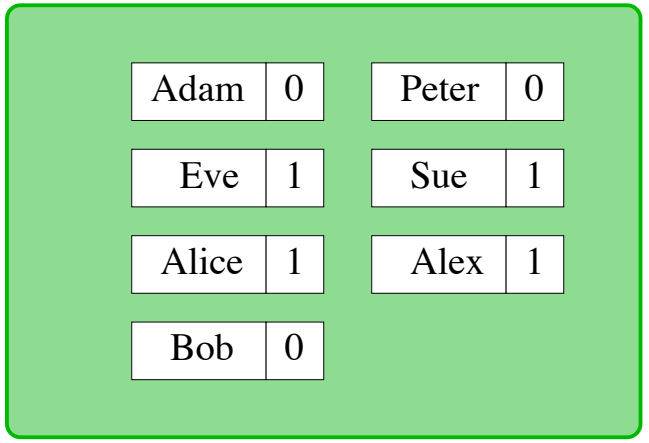


Retrieval

$S \subseteq U$.

Store $f : S \to R$, where w.l.o.g. $R = \{0,1\}^r$,

in data structure D_{retr}.



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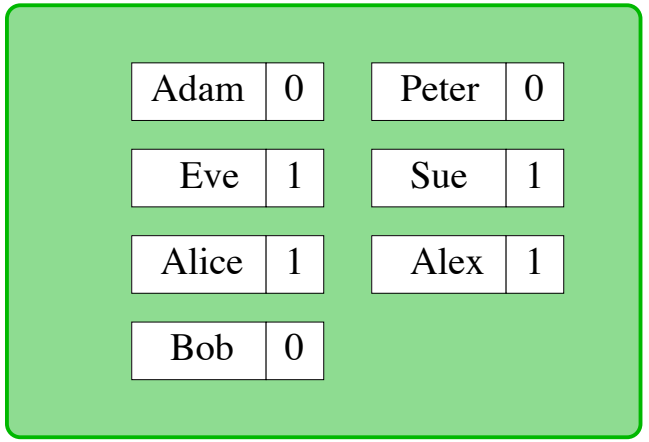

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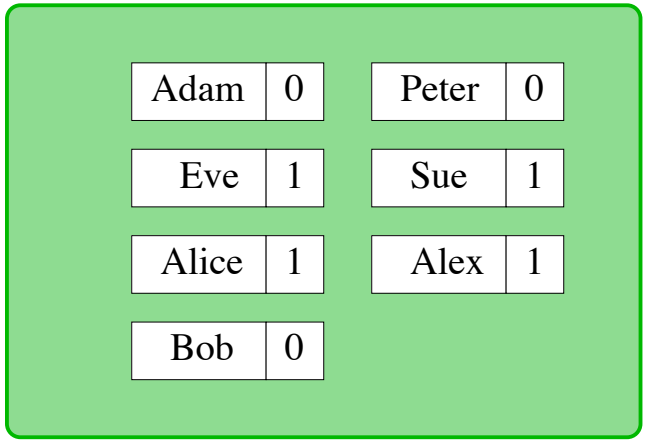

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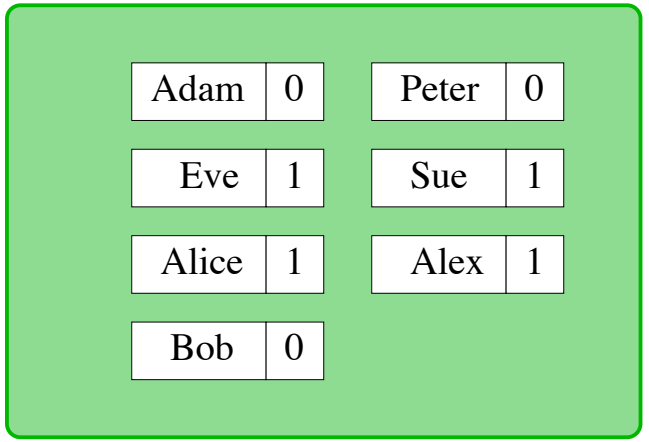

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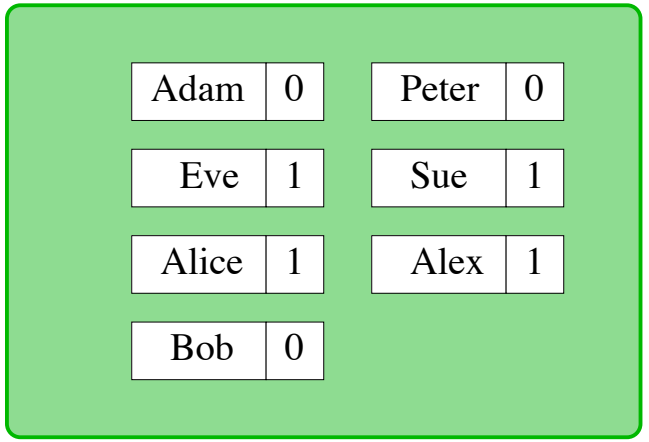

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- Evaluation time $O(1)$ (unless said otherwise).

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Assumption: Fully random hash functions for free.

Overview

- Introduction, Overview
- Basic approach: Hash-read-add
- Previous work
- Results
- Calkin's theorem on sparse random matrices
- Retrieval structure I
- Approximate membership
- Retrieval structure II: Construction in linear time
- Multiple-choice hashing and retrieval structures
- Summary

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Data structure: Array $T[0\,..\,m-1]$, entries from $R = \{0,1\}^r$.

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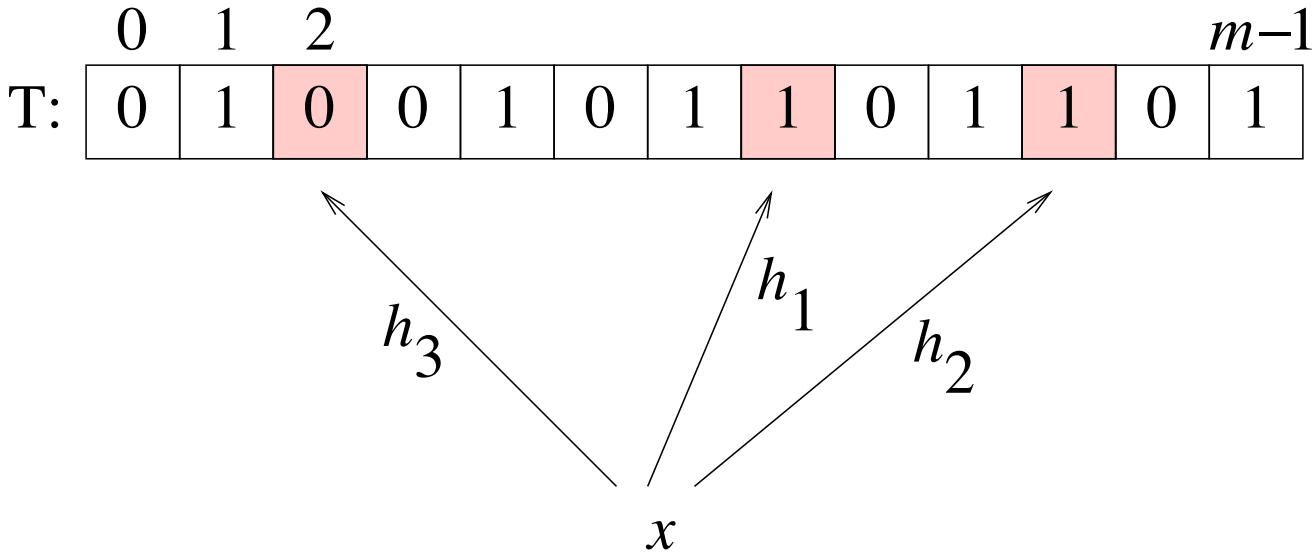

$$f(x) = \bigoplus_{1 \le \ell \le k} T[h_\ell(x)]$$

Evaluation time: $O(k)$. Nonadaptive reads!

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Sufficient condition for this to work:

Matrix $M_A = (p_{ij})_{\substack{1 \le i \le n \\ 0 \le j < m}} =$ with $p_{ij} = \begin{cases} 1 & \text{, if } j \in A_{x_i}, \\ 0 & \text{, otherwise,} \end{cases}$

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	0	1	2	3	4	5	6	7
$A(x_1)$:	0	1	0	0	1	0	0	1
$A(x_2)$:	1	0	1	0	0	0	1	0
$A(x_3)$:	0	1	0	0	1	0	1	0
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Alternative:

$R \subseteq \mathrm{GF}(q)$, any finite field (like $\mathbb{Z}_p$), calculate in $\mathrm{GF}(q)$.

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[3] [Chazelle, Kilian, Rubinfeld, Tal 2004]

Implicit in work on "Bloomier filter", $r \geq 1$, $m = O(n)$.

Ad-hoc analysis.

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M_A can be brought into **echelon form** by row and column **exchanges**.

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Thresholds: (Acyclicity whp **if** $m \geq (1+\gamma)n > \gamma_k n$):

k	2	3	4	5	6	asympt.
γ_k	2	1.222	1.295	1.425	1.570	$\ln k$

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Advantage: Solve linear system in time $O(nk)$.

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k	2	3	4	5	6	asympt.
β_k^{-1}	2	1.1243	1.034	1.011	1.0038	$1 + e^{-k}/\ln 2$
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A retrieval data structure using the hash-read-add scheme with k accesses for a query and space $(1+\gamma)rn$ bits can be built if $1+\gamma > \beta_k^{-1} \approx 1 + e^{-k}/\ln 2$.
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Proof: Theorem 1 plus "splitting".

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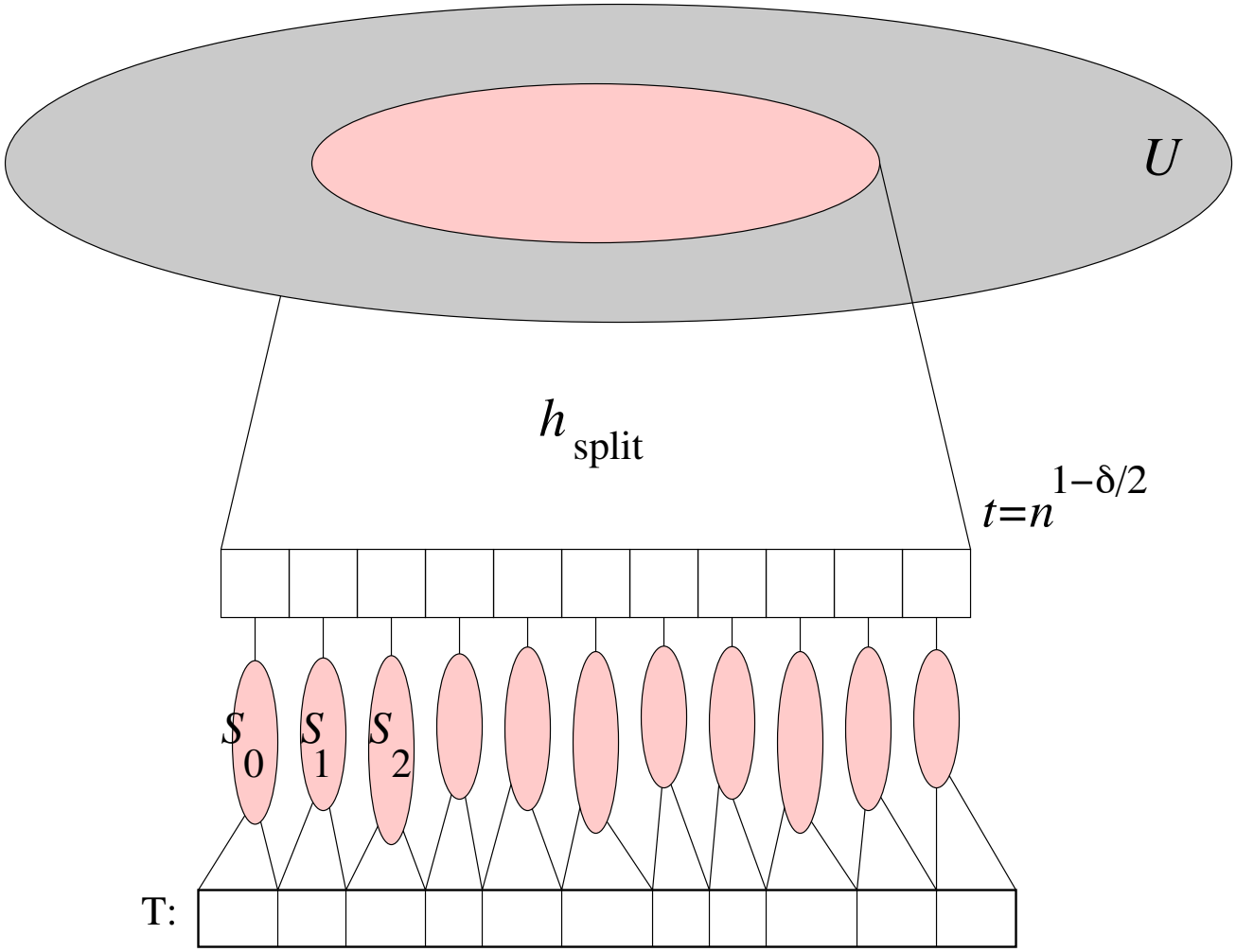

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Retrieval for y: access retrieval data structure for $S_{h_{\text{split}}(y)}$.

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Space: $nr + O(n)$ bits.

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General construction:

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Assume **any** algorithm for retrieval structure for range $R = \{0,1\}^r$

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Construction time $O(n^3)$ resp. $O(n^{1+\delta})$.

Retrieval: Construction in linear time

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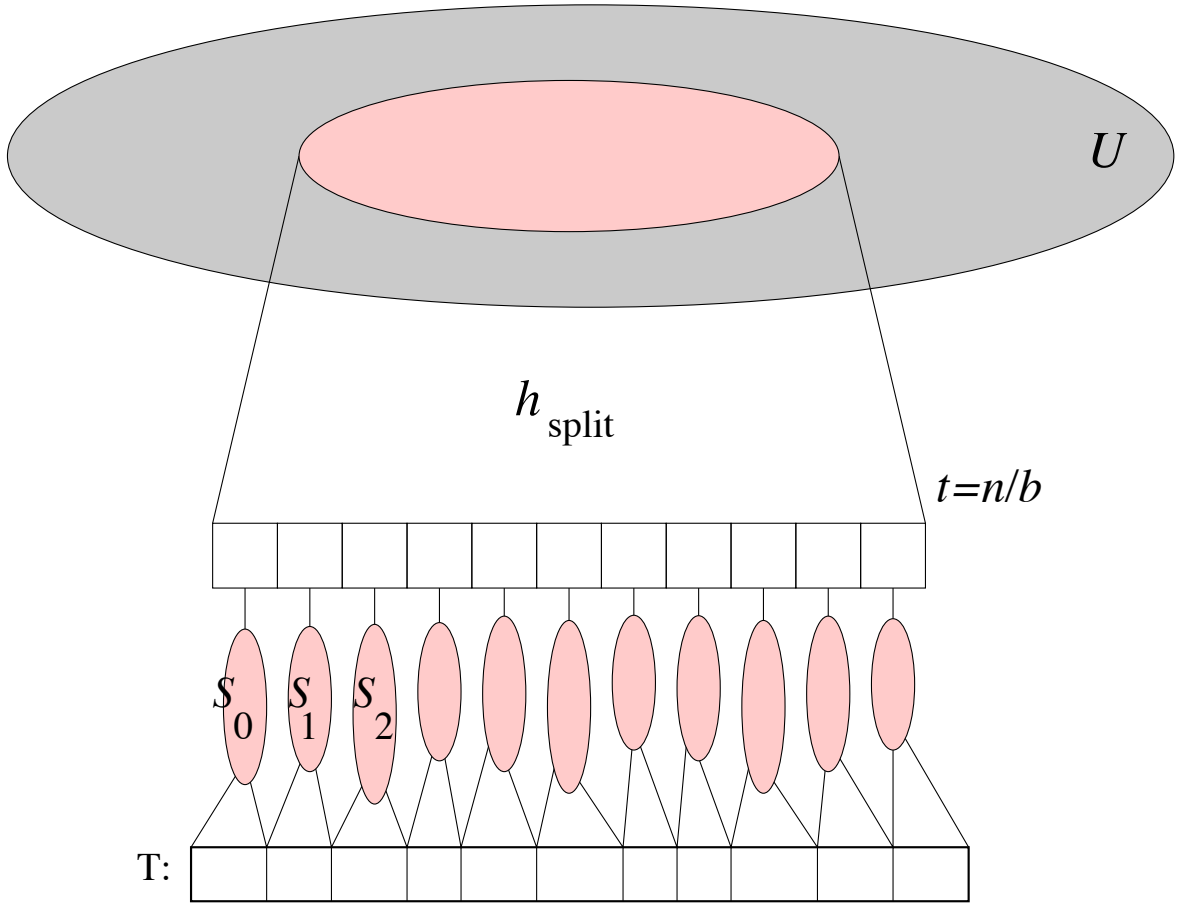
Extra level of hashing: $h_{\text{split}} : U \to [t]$.



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Done?

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No!

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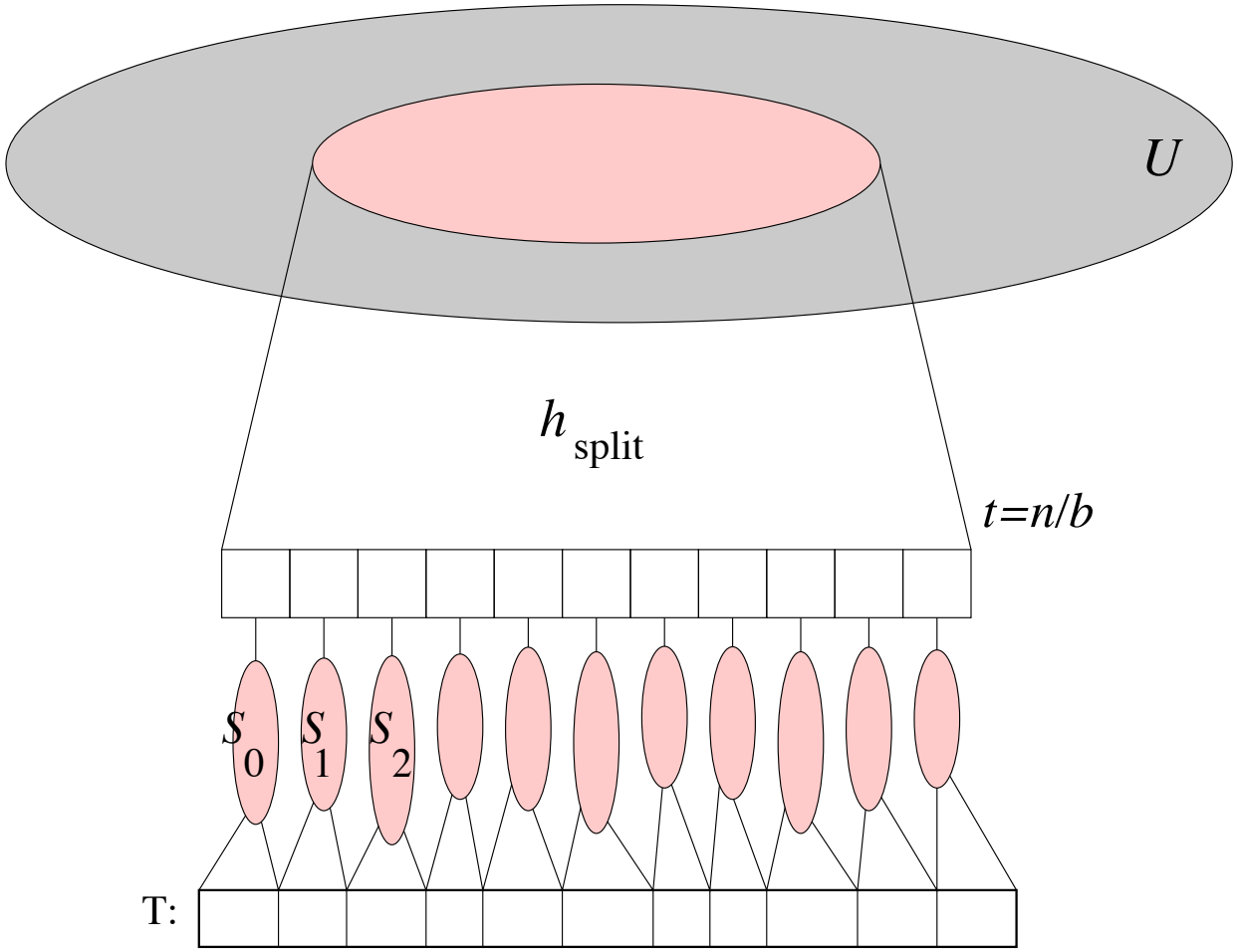
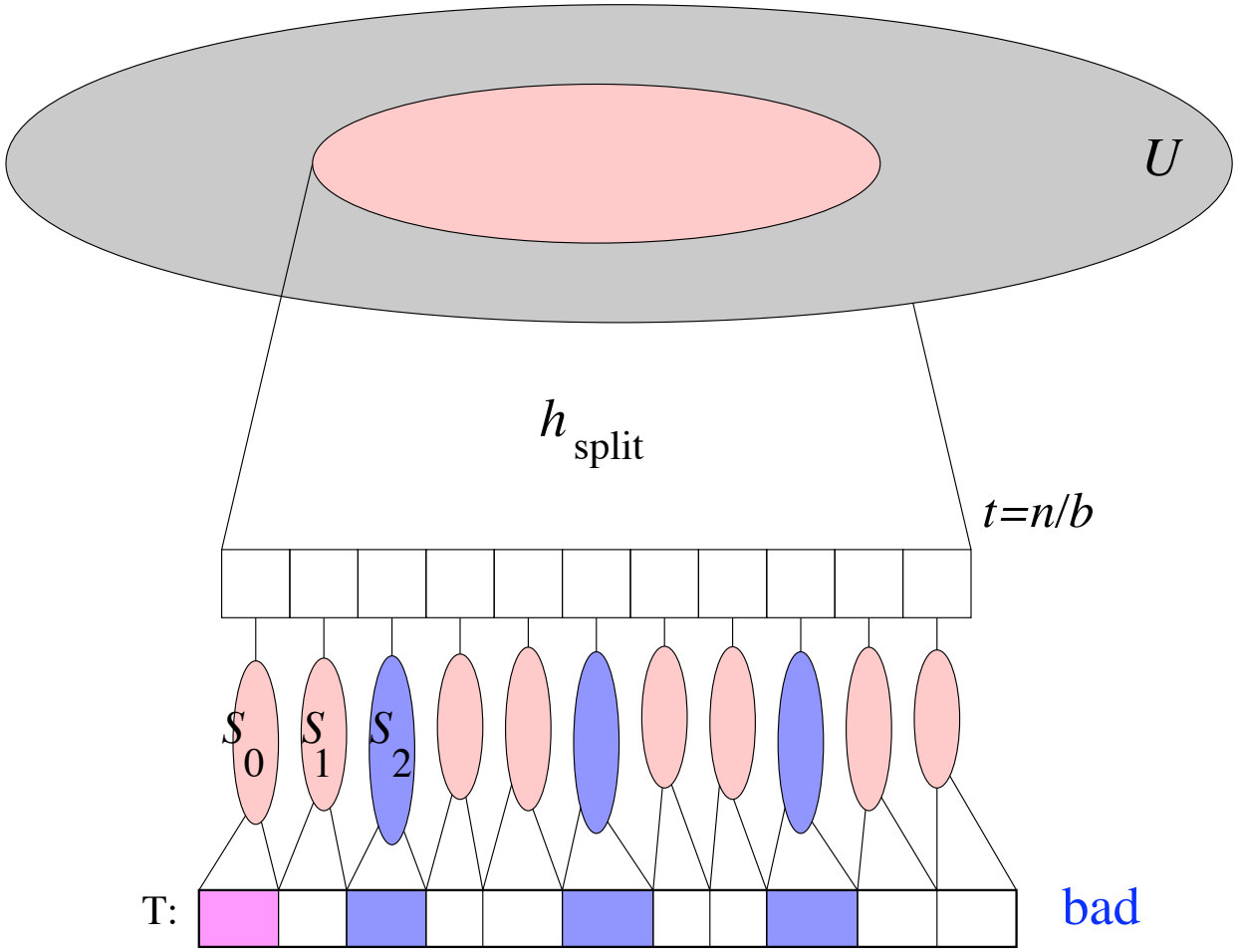
No! "Bad chunks":



Construction in linear time

No! "Bad chunks": (1) Overflow (2) Construction fails.



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Flag: $v[i] = 0$ if chunk i is bad and $v[i] = 1$ otherwise.

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Time $O(k)$. Nonadaptive reads!

Almost optimal space, logarithmic evaluation time

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Multiple-choice hashing and retrieval structures

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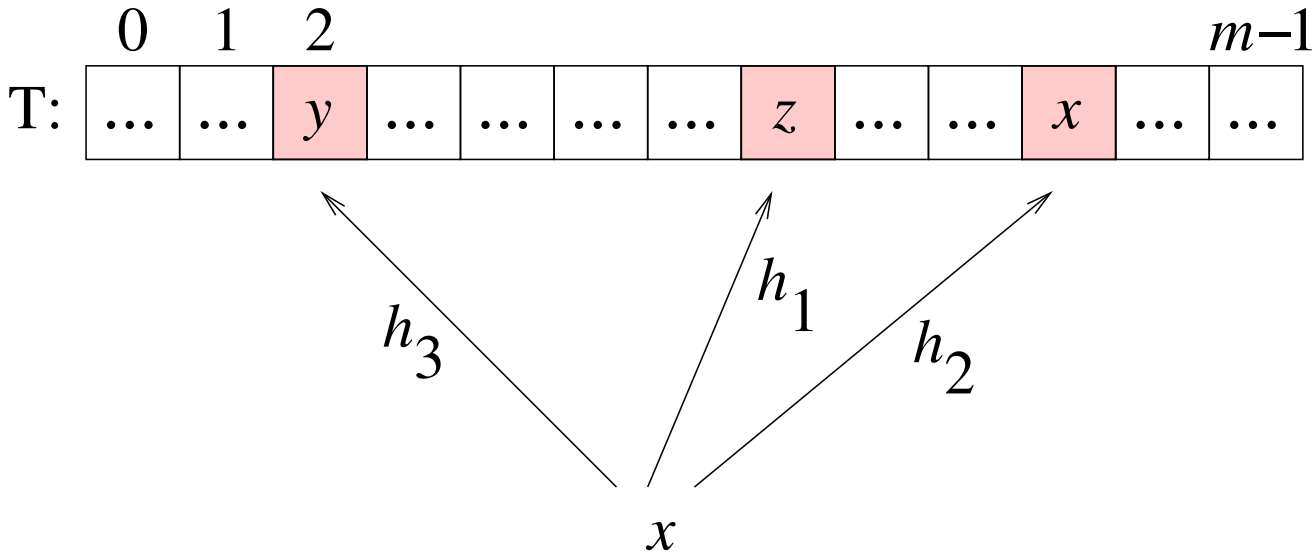
Dictionary structure with multiple choices and retrieval:



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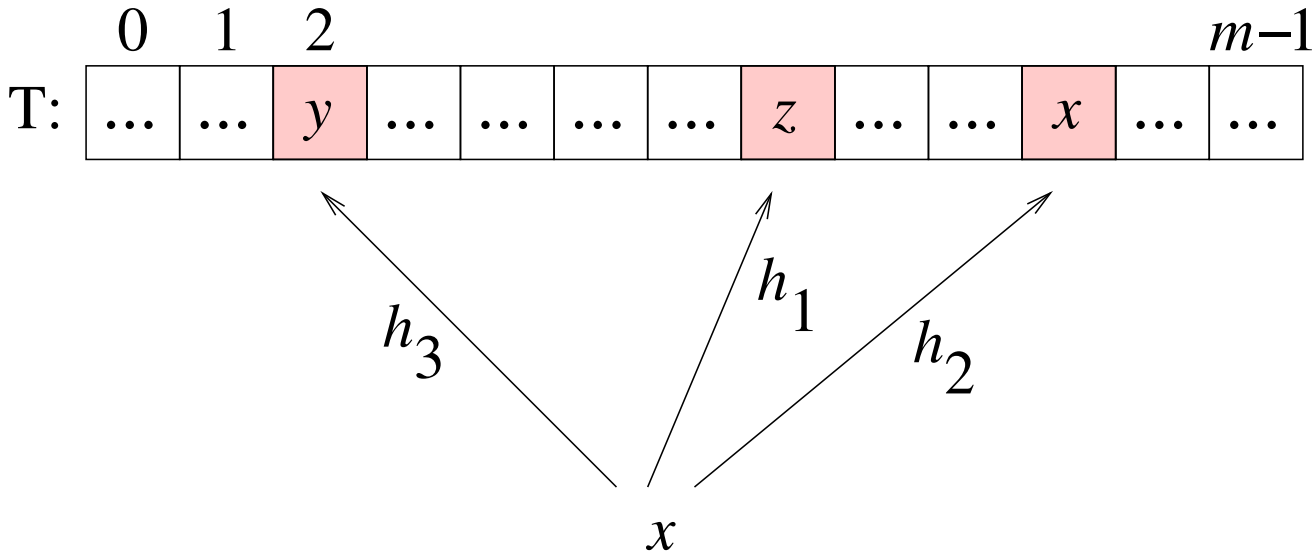

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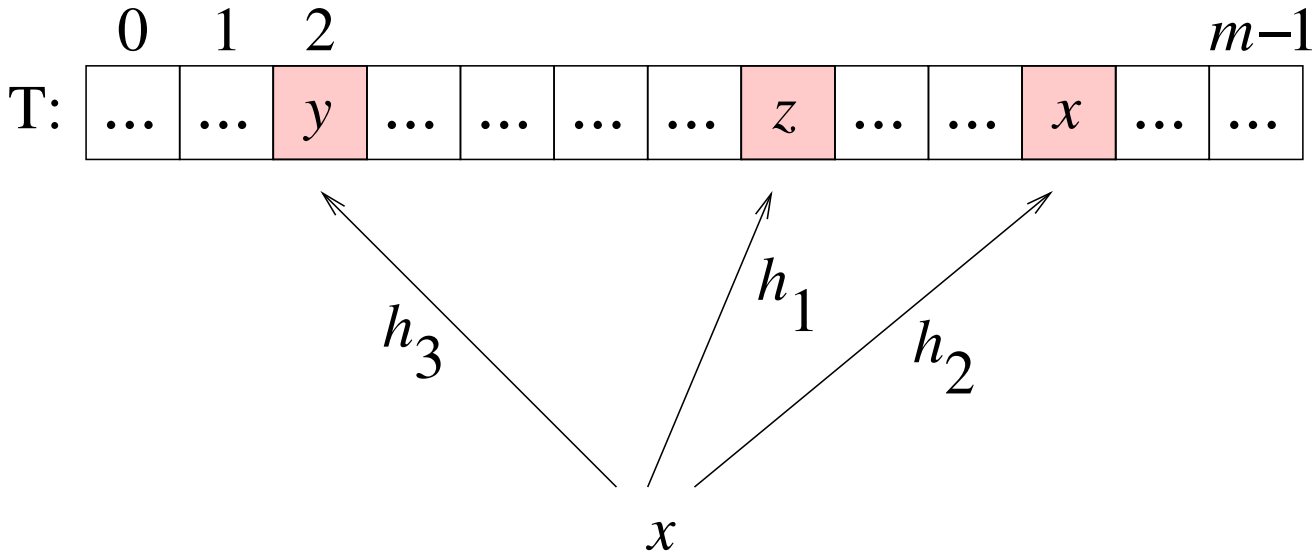

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Many variants: ". . . cuckoo hashing", "balanced allocation".

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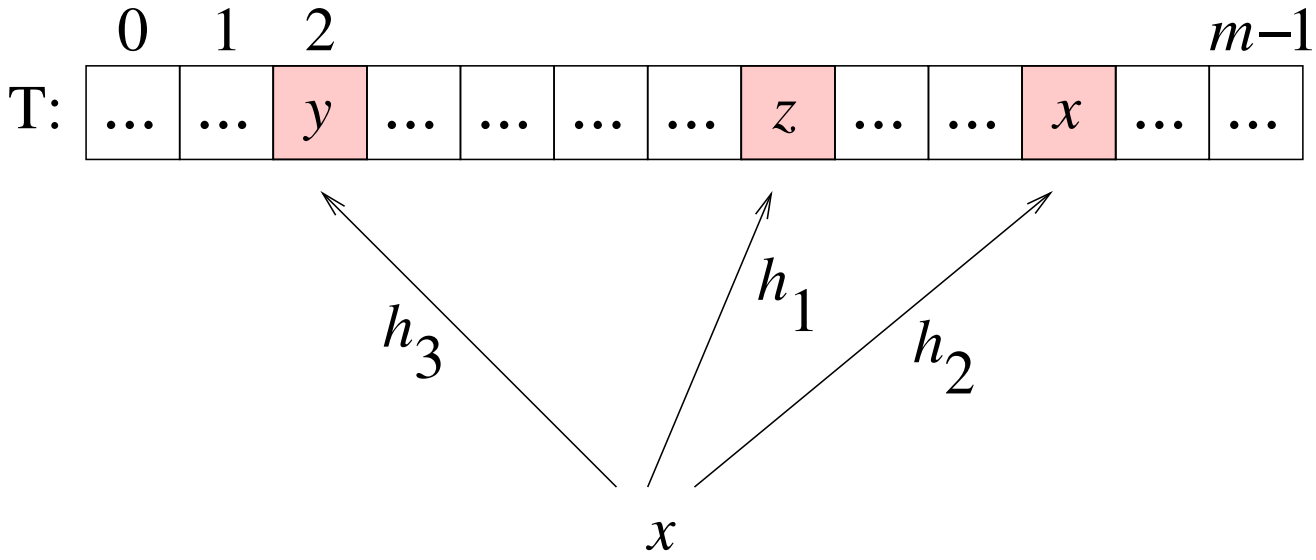

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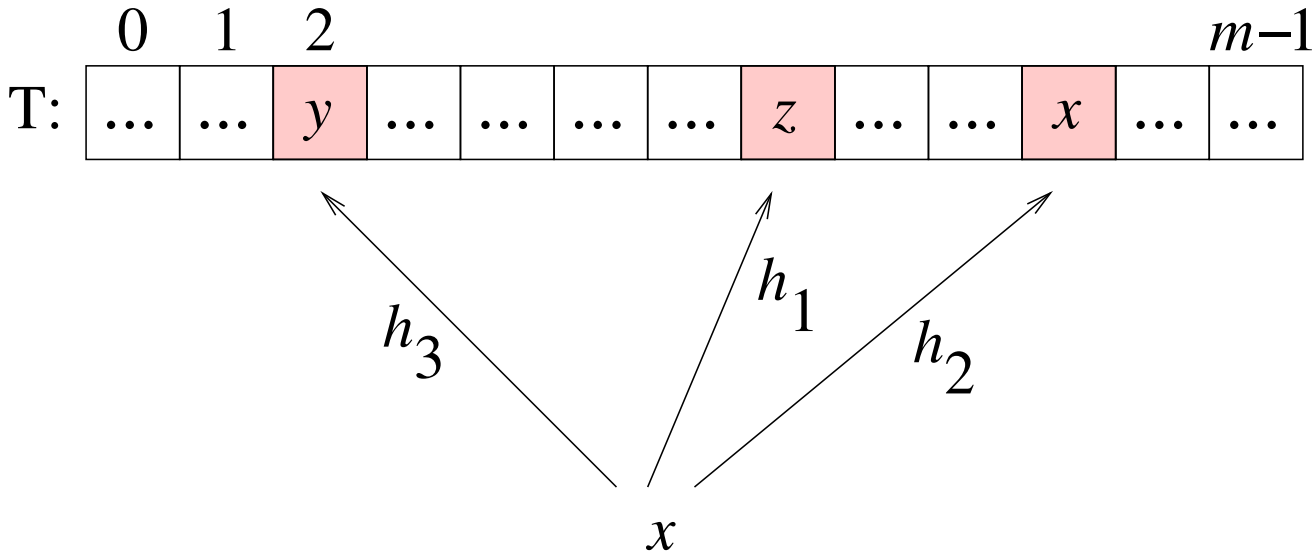

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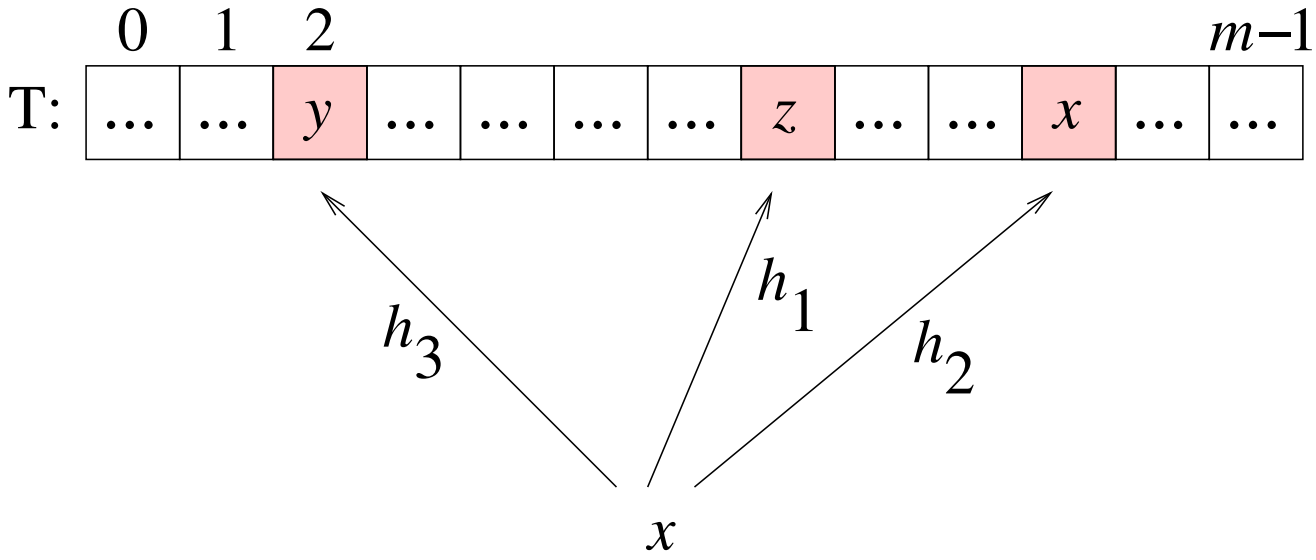

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(. . . if and only if one can store x_i in a slot in $A(x_i)$, without collision).

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Observation 1:

M_A has full row rank $\Rightarrow$ A is suitable for S.

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Reason: M_A has a regular square submatrix M'_A.

	$[m]$							
	0	1	2	3	4	5	6	7
$A(x_1)$:	0	1	0	0	1	0	0	1
$A(x_2)$:	1	0	1	0	0	0	1	0
$A(x_3)$:	0	1	0	0	1	0	1	0
$A(x_4)$:	0	1	1	1	0	0	0	0
$A(x_5)$:	1	0	1	0	0	1	0	0

$$\det(M'_A) = \sum_{\pi \in S_n} \operatorname{sign}(\pi) \prod_{1 \le i \le n} p_{i\pi(i)} \neq 0.$$

Multiple-choice hashing and retrieval structures

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(Tiny improvement of space bound of [Fotakis et al., 2005], d-ary cuckoo hashing. — Conjecture: Optimal.)

Multiple-choice hashing and retrieval structures

Observation 2: A suitable for S $\Rightarrow$

	$[m]$							
	0	1	2	3	4	5	6	7
$A(x_1)$:	0	1	0	0	1	0	0	1
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M_A has a square submatrix M'_A with a nonvanishing term $\mathrm{sign}(\pi) \prod_{1 \le i \le n} p_{i\pi(i)}$ in the determinant . . .

Multiple-choice hashing and retrieval structures

Observation 2: A suitable for S $\Rightarrow$

	$[m]$							
	0	1	2	3	4	5	6	7
$A(x_1)$:	0	8	0	0	4	0	0	5
$A(x_2)$:	0	6	9	0	0	0	2	0
$A(x_3)$:	0	9	8	0	0	0	5	0
$A(x_4)$:	0	7	7	4	0	0	0	0
$A(x_5)$:	5	0	2	0	0	7	0	0

If we replace the 1's in M_A with random elements of $\mathbb{F} - \{0\}$ (finite field $\mathbb{F}$), yielding $M_A(\mathbb{F})$, **then** $M_A(\mathbb{F})$ has full row rank with probability $\geq 1 - \frac{n}{|\mathbb{F}|}$ (Schwartz-Zippel Theorem).

Multiple-choice hashing and retrieval structures

Resulting retrieval function ($g_1(x), \ldots, g_k(x)$ are random hash values, calculate in $\mathbb{F}$):

$$f(x) = \sum_{1 \leq \ell \leq k} g_\ell(x_i) \cdot T[h_\ell(x)]$$

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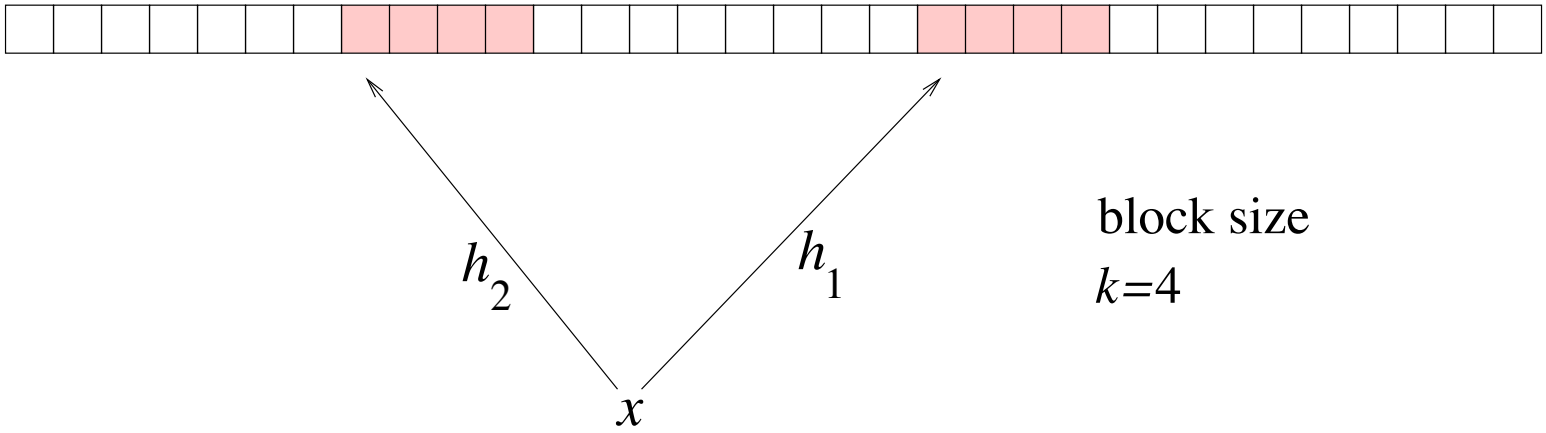

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Multiple-choice hashing and retrieval structures

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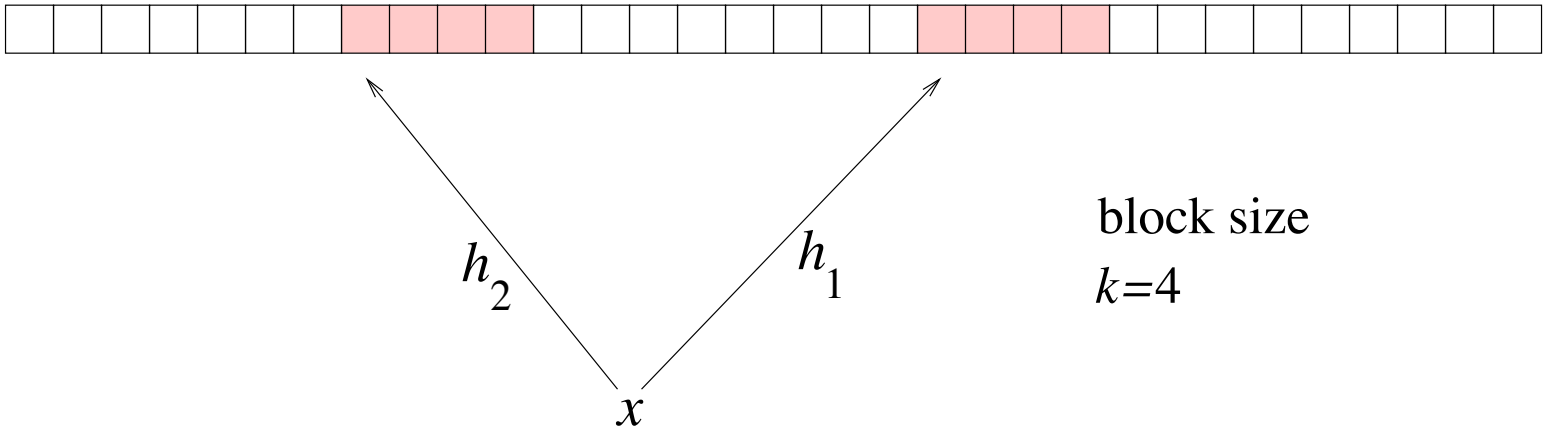

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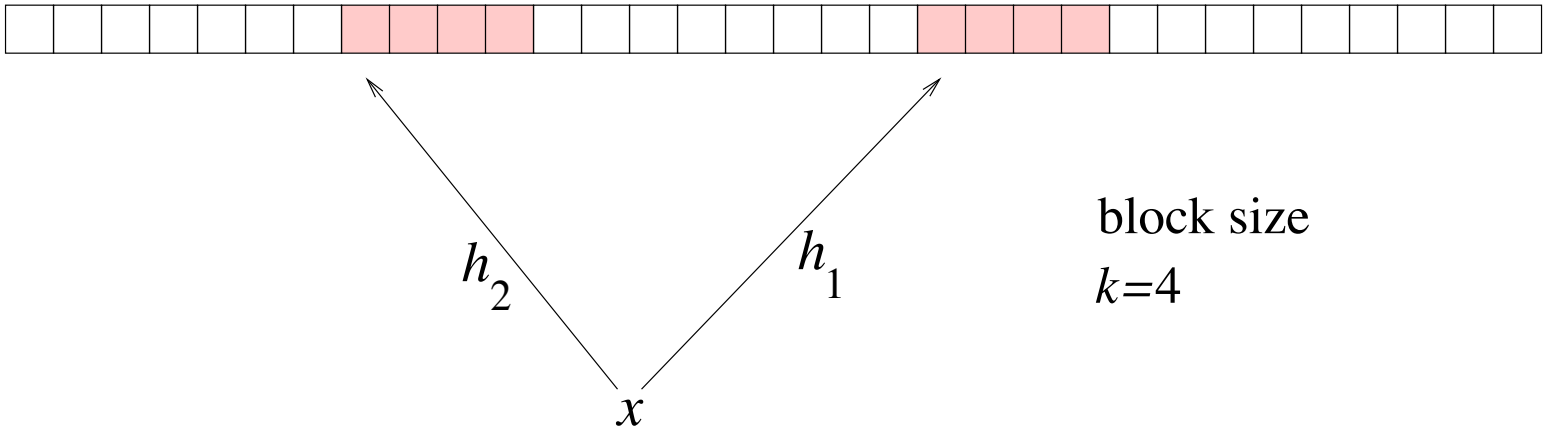

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[D.,Weidling 2007] Read $2 \times k$ locations, space $(1 + e^{-ck})nr$.

[Czumaj *et al.* 2003] Read $2 \times O(\log n)$ locations, space nr.

Summary

- Retrieval in time $O(k)$, space $(1 + e^{-k})nr$.
- Construction time
  $O(n^{1+\delta})$ (fair) or $O(n)$ (contrived, asymptotic)
- Close connection retrieval $\leftrightarrow$ approximate membership
- Close connection retrieval $\leftrightarrow$ hash table à la cuckoo
- Cache-friendly retrieval/approximate membership structures with almost optimal space
  (range size $\geq n^{\delta}$ or error bound $\leq 1/n^{\delta}$.)

Thank you!