Basic External Memory Data Structures

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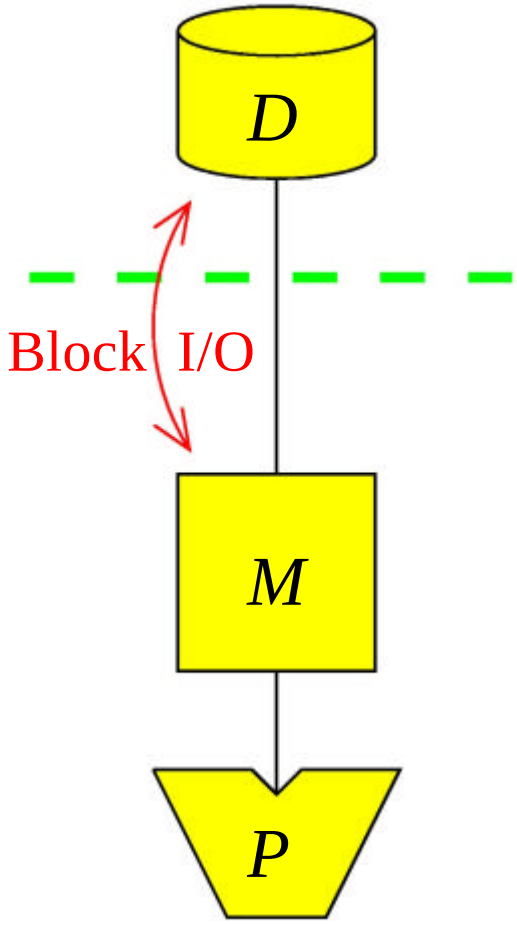
External Memory Model



In this talk, focus is on transfers between *two levels* of the memory hierarchy (I/O model)

N = # of items in the problem instance

B = # of items per external memory block

M = # of items that fit in internal memory

Z = # of items in output

We assume (for convenience) that $M > B^2$

Overview of Talk

Part I. Elementary data structures

- Data structures: Linked lists, stacks and queues
- Concepts: Locality (grouping), split-fuse, buffering

Part II. Search trees

- Data structures: B-trees, buffer trees
- Concepts: Balancing, output sensitivity, bootstrapping, batched dynamic problems

Part III. Non-comparison-based data structures.

- Data structures: Dictionaries, predecessor d.s., range dictionaries
- Concepts: Hashing, word length dependence

Part I. Elementary Data Structures

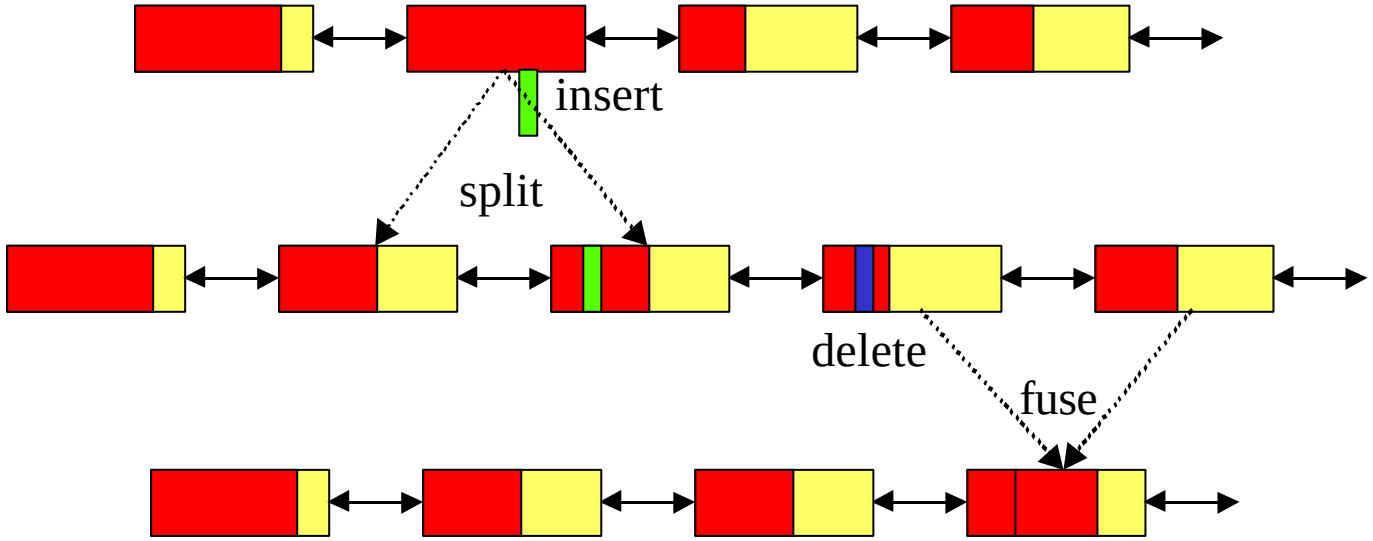
Linked Lists

- Usual internal memory data structures provide insertions and deletions in linked lists in *O(1)* I/Os
- Like in most internal pointer structures, there is no guarantee of *locality*, i.e., that data items needed around the same time resides close to each other in memory
- This means that traversal of Z elements may take *O(Z)* I/Os rather than the optimal bound of *O(1+Z/B)* I/Os for scanning a list

Optimal External Memory Linked Lists

Approach:

- To achieve locality, *group* adjacent elements in chunks of size $Q(B)$, stored in linked blocks
- Maintain chunk size by *splitting* and *fusing* chunks



Stacks and Queues

- Naïve externalization gives *O(1)* I/O per operation
- Much better performance can be achieved by *buffering*

Buffered queue:

- "Out-buffer" stores the $k£ B$ latest elements to enter the queue
- "In-buffer" stores the $k'£ B$ next elements to leave the queue
- Flushing out-buffer and filling in-buffer can be done in *1* I/O
  $\Rightarrow$ *O(1/B)* I/O amortized per operation

Buffered stack:

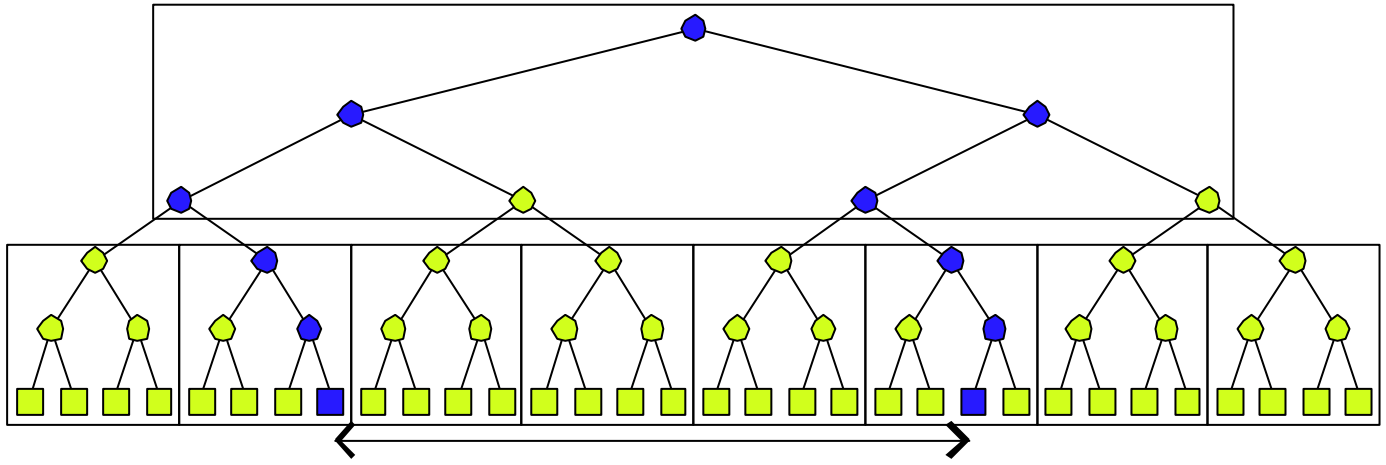
- Similar to the buffered queue
- *O(1/B)* I/O amortized per operation

Part II. Search Trees

B-trees

[Bayer-McCreight ‘72, Huddleston-Mehlhorn ‘82]

- Storing binary trees arbitrarily in external memory gives $O(\log_2 N)$ I/Os per query/update
- B-trees block nodes together such that each block can be stored in one external memory block
- The height of the tree (in blocks) becomes $O(\log_B N)$.



B-tree Summary

B-tree properties:

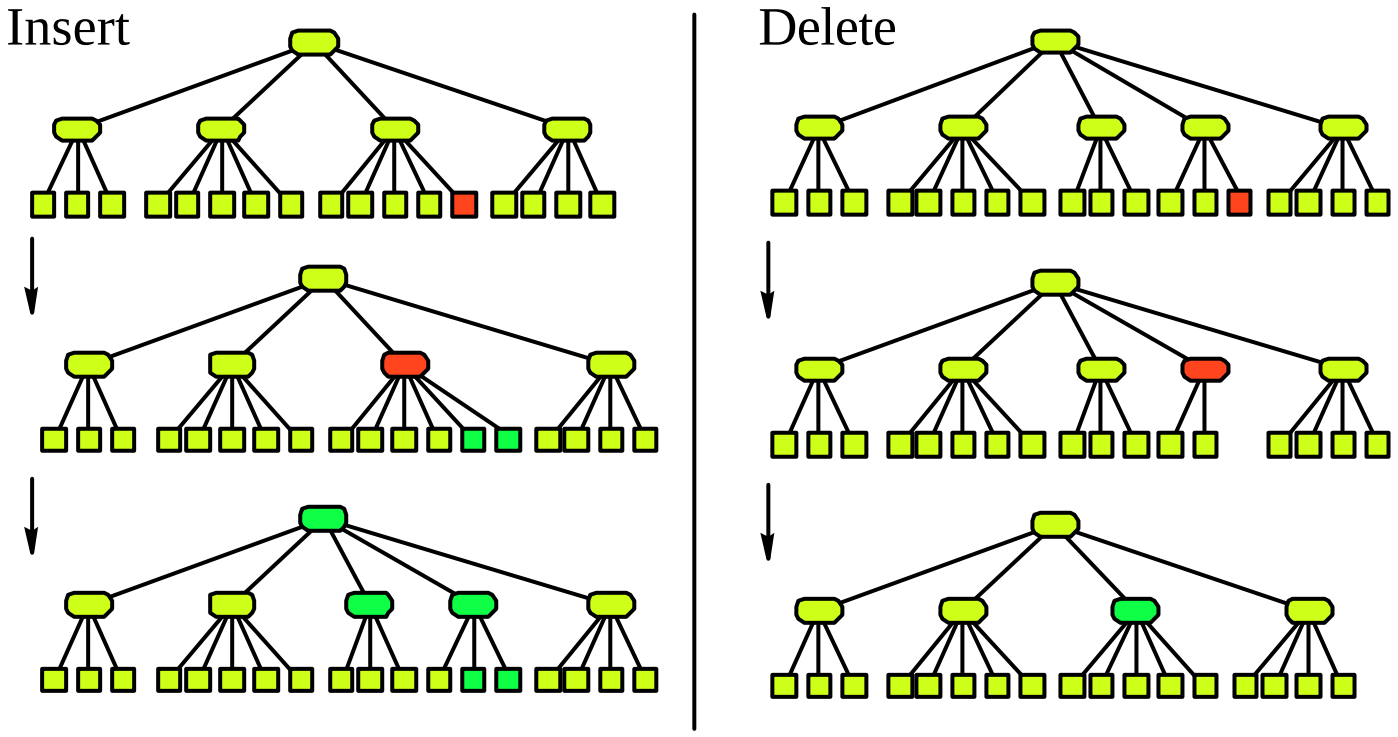
- All leaves – consisting of $\Theta(B)$ elements - on the same level
- Internal nodes have degree $\Theta(B)$ (except possibly the root)
- Depth is $O(\log_B N)$
- Uses $O(N/B)$ space (optimal)

Queries:

- Find a key, or the nearest neighbor of a key, in $O(\log_B N)$ I/Os
- Find all Z keys in a range $[a;b]$ (a "range search") in $O(\log_B N+Z/B)$ I/Os (output sensitive!)
- Time for queries is optimal for *comparison-based* algorithms.

Updates in B-trees

- Blocking hard to maintain by rotations as used in internal memory
- Rebalancing instead uses split/fuse (and share):



$\Rightarrow O(\log_B N)$ update bound

B-tree Applications

B-trees are used for a wide range of applications in computational geometry, graph algorithms, string processing, etc.

(many such applications to come in the next talks)

General goal:

- Search a database of N objects in $O(\log_B N+Z/B)$ I/Os
- Support updates in $O(\log_B N)$ I/Os

Output Sensitivity and Bootstrapping

Output sensitivity problem:

- Must be sure to report $W(B)$ elements for each I/O "after search".
- A *secondary structure* is often attached to each B-tree node to help reporting elements from subtrees with few elements to report
- When a node splits or fuses, the secondary structure must be rebuilt, which often involves traversing *the entire subtree*

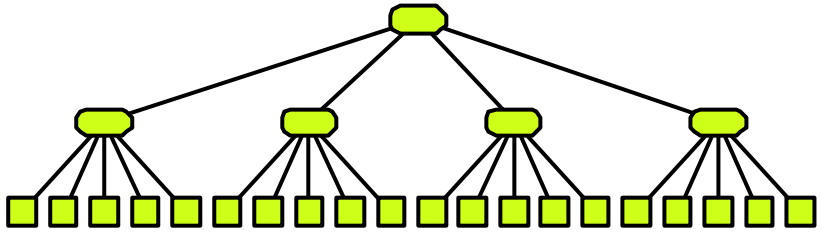
Bootstrapping:

- The secondary structure often solves a small version of the problem to be solved, i.e., it is used to *bootstrap* the main solution

Weight-balanced B-trees

Problem:

- Ordinary B-tree nodes can split/fuse often, so rebuilding the secondary structure can be expensive.



Weight-balanced B-tree:

- Like B-tree but with *weight* instead of degree constraint
- Balanced with split/fuse as B-tree
  - node ν only splits/fuses for every $\Omega(\omega(\nu))$ updates below it
  - amortized $O(\log_B N)$ I/O update if the secondary structure can be rebuilt in $O(\omega(\nu))$ I/Os

Persistent B-trees

- In database applications we are often interested in being able to access previous versions of database
- Partial persistence:
  - Update current version, query all versions
- Partially persistent B-tree (multi-version B-tree) can be obtained using standard techniques [DSST '89, BGOSW '96, VV '97]
  - $O(\log_B N)$ update, $O(\log_B N+Z/B)$ query, $O(N/B)$ space
  - N is total number of operations performed
- Idea:
  - Elements and nodes augmented with existence intervals
  - Maintain that every node contains $\Theta(B)$ alive elements in its existence interval

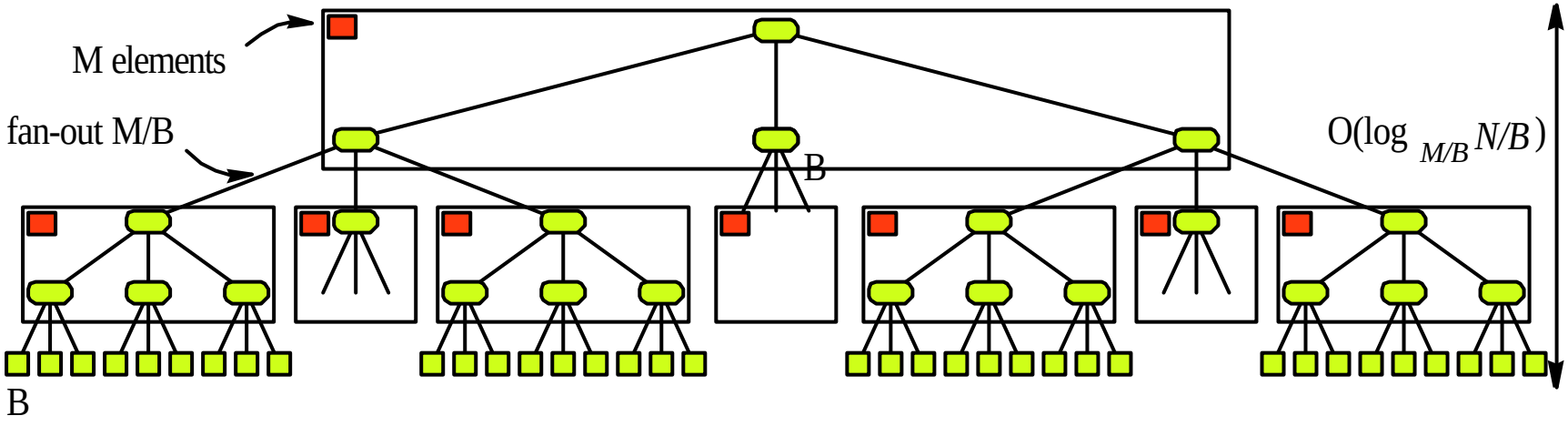
B-tree Construction

- Insertion in $O(\log_B N)$ I/Os in B-tree, weight-balanced B-tree, and persistent B-tree
- Sorting N elements takes $\Theta(\frac{N}{B}\log_{M/B}\frac{N}{B})$ I/Os
- Construction by repeated insertion use $O(N\log_B N)$ I/Os
  - more than a factor B sub-optimal

- We can build B-tree and weight-balanced B-tree in $\Theta(\frac{N}{B}\log_{M/B}\frac{N}{B})$ I/Os bottom-up
  - not so clear with persistent B-tree
- We need $O(\frac{\log_{M/B}\frac{N}{B}}{B})$ I/Os per operation to be optimal

Buffer Tree

[Arge '95]



- Main idea: Logically group nodes together and add buffers
  - Insertions done in a "lazy" way — elements inserted in buffers.
  - When a buffer runs full elements are pushed one level down.
  - Buffer-emptying in $O(M/B)$ I/Os

    $\Rightarrow$ every *block* touched constant number of times on each level

    $\Rightarrow$ inserting N elements (N/B blocks) costs $O(\frac{M}{B} \log_{M/B} \frac{N}{B})$ I/Os.

Buffer Tree Summary

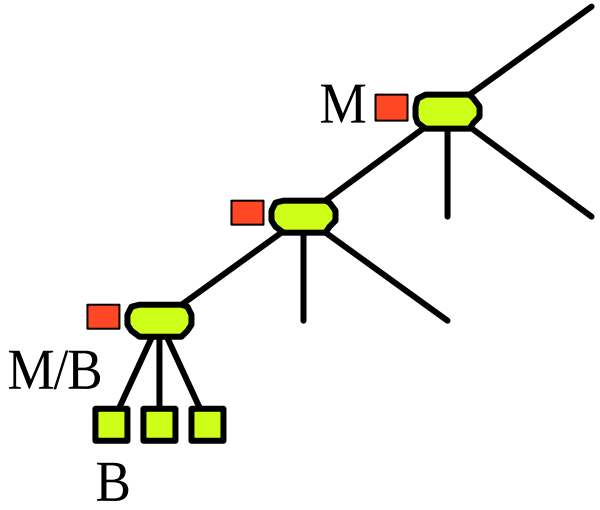
- Buffer tree can be used in $O(\frac{N}{B}\log_{M/B}\frac{N}{B})$ sorting algorithm
- Deletes and range queries can be handled similarly to insertions
  - Queries become *batched*, i.e., answers are not given right away.
- Batched dynamic problems occur in many applications, in particular "sweepline" algorithms in computational geometry.

Buffer Tree Priority Queue

Buffer trees also yield an optimal comparison-based priority queue.

Approach:

- Empty all buffers on leftmost path
  $\Rightarrow O(\frac{M}{B} \log_{M/B} \frac{N}{B})$ I/Os
- Delete M elements in leftmost leaves
- Next M deletemins free $\Rightarrow O(\frac{\log_{M/B} \frac{N}{B}}{B})$ I/Os amortized



A "worst-case efficient" priority queue also exists [BK '98].

Implementations

Libraries:

- LEDA-SM (www.mpi-sb.de/~crauser/leda-sm)
- TPIE (www.cs.duke.edu/TPIE)

Implementations of basic external memory data structures:

- B-trees — LEDA-SM, TPIE
- Persistent B-trees — TPIE
- Buffer trees — LEDA-SM
- Priority queues — LEDA-SM, TPIE

Part III. Non-comparison-based Data Structures

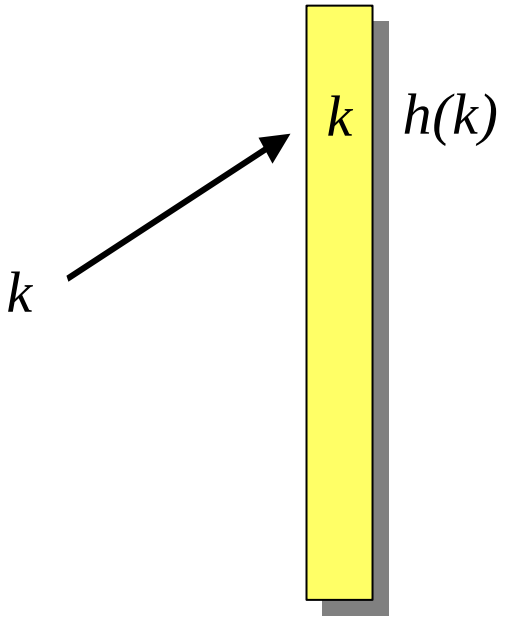
Dictionaries and Classic Hashing

Hashing in a nutshell:

- Use a function *h* to guide searches
- Search for key *k* (starting) at position *h(k)* in a ”hash table”

Intuition:

If *h* behaves ”randomly”, it distributes the keys evenly throughout the table



Collision resolution:

– Mechanism for dealing with keys hashing to the same location

– Easier* in external memory, as each block can accommodate *B* keys.

External Performance of Classic Hashing

Let α = *load factor* (fraction of hash table occupied by keys)

- Hash table consists of $N/(B\alpha)$ blocks
- The probability that a given block overflows is at most

$$e^{-B(1/\text{á}-1)^2/3}$$

- *No block overflows* unless N is exponentially larger than B (w.h.p.)

Alternative collision resolution:

– Store elements from overflowing blocks in internal memory.

– Guaranteed 1 I/O dictionary lookup

Two-way Hashing

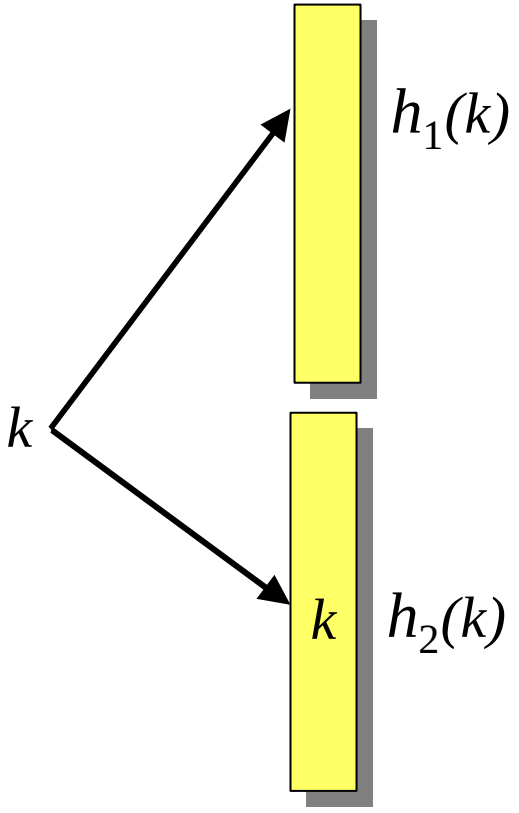
[ABKU '99, BCSV '00]

If two "banks" of external memory can be accessed simultaneously, better "1 I/O" probability bounds can be achieved

Two-way Hashing:

- Use *two* hash functions h_1 and h_2 with disjoint ranges
- Insert new key k in the *least full* of block $h_1(k)$ and $h_2(k)$
- The probability that a given block overflows is:

$$2^{-2^{\Omega((1-\alpha)B)}}$$



Resizing Hash Tables

Maintaining a (high) load factor:

- When the size of the key set changes, so must the hash table size
- Rehashing should be easy, ideally done in a single linear scan

Using a ”mother” hash function:

- Use hash function φ with large range to derive other hash functions
- Poor choice: $x \mapsto \varphi(x) \bmod r$
- Good choice: $x \mapsto \varphi(x) \operatorname{div} (R \operatorname{div} r)$, where R = size of φ’s range
  - The ordering of hash function values never changes

Other Non-comparison-based Data Structures

Dictionary generalizations:

- **Predecessor dictionary:**
  Given k find the largest key j in the dictionary with $j<k$ (if any)
- **Range dictionary:**
  Given a, b find all keys j in the dictionary with $j \in [a\ ;\ b]$

Techniques:

- Use and manipulate bit representation of keys (w bits)
- Recurse on smaller set *or shorter keys*
- Use dictionaries to look up ”pre-computed” answers

Predecessor Dictionary

There is a predecessor dictionary [A'96,BF'99,AT'00] that uses linear space and, for N keys of w bits supports updates and queries in time

$$O\left(\min\left(\sqrt{\log N / \log\log N}, \log_{Bw} N, \frac{\log w}{\log\log w}\log\log_{Bw} N\right)\right)$$

Any predecessor dictionary using space poly(n) must have query time

$$\Omega\left(\min\left(\log_{Bw} N, \frac{\log w}{\log\log w}\right)\right) \qquad \text{[BF '99]}$$

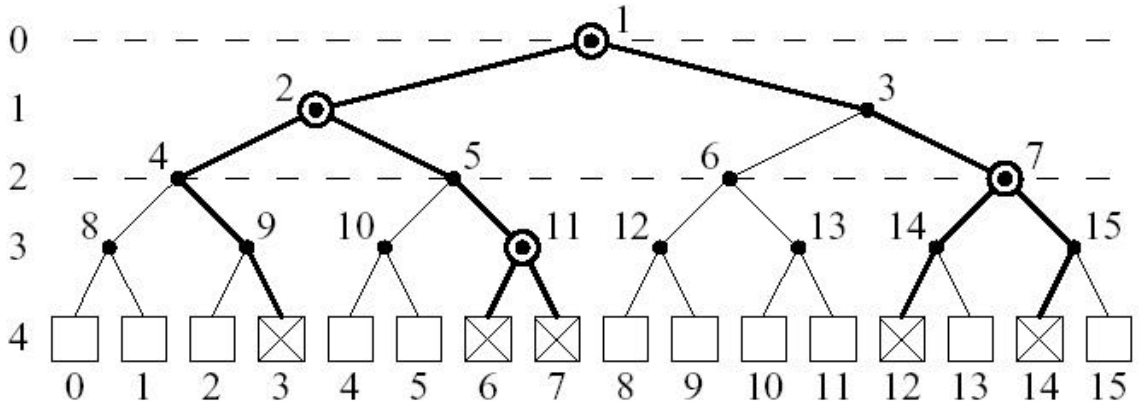
The best possible query time in terms of N is:

$$\Omega\left(\min\left(\sqrt{\log N / \log\log N}, \log_B N\right)\right)$$

Range Dictionary

Main observations:

- We report only keys that share the longest common prefix of *a* and *b*
- The trie of *N* *w*-bit strings can be stored such that the longest prefix of any string can be found in constant time. Main issue: Space



Result:

- Range queries in $O(1+Z/B)$ I/Os and space $O(N)$
- Static, no efficient dynamization known (yet)

Summary

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